Metastability of Potential Games

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Abstract

One of the main criticisms to game theory concerns the assumption of full rationality. Logit dynamics is a decentralized algorithm in which a level of irrationality (a.k.a. “noise”) is introduced in players’ behavior. In this context, the solution concept of interest becomes the logit equilibrium, as opposed to Nash equilibria. Logit equilibria are distributions over strategy profiles that possess several nice properties, including existence and uniqueness. However, there are games in which their computation may take exponential time. We therefore look at an approximate version of logit equilibria, called metastable distributions, introduced by Auletta et al. [4]. These are distributions which remain stable (i.e., players do not go too far from it) for a super-polynomial number of steps (rather than forever, as for logit equilibria). The hope is that these distributions exist and can be reached quickly by logit dynamics.

We devise a sufficient condition for potential games to admit distributions which are metastable no matter the level of noise present in the system, and the starting profile of the dynamics. These distributions can be quickly reached if the rationality level is not too big when compared to the inverse of the maximum difference in potential. Our proofs build on results which may be of independent interest. Namely, we prove some spectral characterizations of the transition matrix defined by logit dynamics for generic games and relate several convergence measures for Markov chains.

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1 Introduction

One of the most prominent assumptions in game theory dictates that people are rational. This is contrasted by many concrete instances of people making irrational choices in certain strategic situations, such as stock markets [28]. This might be due to the incapacity of exactly determining one’s own utilities: the strategic game is played with utilities perturbed by some noise.

Logit dynamics [6] incorporates this noise in players’ actions and then is advocated to be a good model for people behavior. More in detail, logit dynamics features a rationality level $\beta \geq 0$ (equivalently, a noise level $1/\beta$) and each player is assumed to play a strategy with a probability which is proportional to her corresponding utility and $\beta$. So the higher $\beta$ is, the less noise there is and the more rational players are. Logit dynamics can then be seen as a noisy best-response dynamics.

The natural equilibrium concept for logit dynamics is defined by a probability distribution over the pure strategy profiles of the game. Whilst for best-response dynamics pure Nash equilibria are stable states, in logit dynamics there is a chance, which is inversely proportional to $\beta$, that players deviate from such strategy profiles. Pure Nash equilibria are then not an adequate solution concept for this dynamics. However, the random process defined by the logit dynamics can be modeled via an ergodic Markov chain. Stability in Markov chains is represented by the concept of stationary distributions. These distributions, dubbed logit equilibria, are suggested as a suitable solution concept in this context due to their properties [5]. For example, from the results known in Markov chain literature, we know that any game possesses a logit equilibrium and that this equilibrium is unique. The absence of either of these guarantees is often considered a weakness of pure Nash equilibria. Nevertheless, as for Nash equilibria, the computation of logit equilibria may be computationally hard depending on whether the chain mixes rapidly or not [3].

As the hardness of computing Nash equilibria justifies approximate notions of the concept [21, 9], so Auletta et al. [4] look at an approximation of logit equilibria that they call metastable distributions. These distributions aim to describe regularities arising during the transient phase of the dynamics before stationarity has been reached. Indeed, they are distributions that remain stable for a time which is long enough for the observer (in computer science terms, this time is assumed to be super-polynomial) rather than forever. Roughly speaking, the stability of the distributions in this concept is measured in terms of the generations living some historical era, while stationary distributions remain stable throughout all the generations. When the convergence to logit equilibria is too slow, then there are generations which are outlived by the computation of the stationary distribution. For these generations, metastable distributions grant an otherwise impossible descriptive power. (We refer the interested reader to [4] for a complete overview of the rationale of metastability.) It is unclear whether and which strategic games possess these distributions and if logit dynamics quickly reaches them.

The focus of this paper is the study of metastable distributions for the logit dynamics run on the class of potential games [22]. Potential games are an important and widely studied class of games modeling many strategic settings. Each such game satisfies a number of appealing properties, the existence of pure Nash equilibria being one of them. A study of conditions under which potential games have metastable distributions was left open by [4] and assumes particular interest due to the known hardness results, see e.g. [14], which suggest that the computation of pure Nash equilibria for them is an intractable problem, even for centralized algorithms.

Our contribution. Auletta et al. [4] define metastable distributions not in asymptotic terms but on a single Markov chain. However, with this definition, we cannot in general describe the transient phase of the dynamics for certain $n$-player games (see Section 3.1; note also that all results in [4] are of asymptotic nature). To address this issue, we formalize the concept of asymptotic convergence/closeness to a metastable distribution, as a function of the number of players of a game. We then note, via the careful construction of an $n$-player potential game, that not all potential games admit metastable distributions.
(cf. Section 3.1). Therefore, we turn our attention to understanding the conditions under which metastability is indeed possible.

We devise a sufficient property for an $n$-player potential game to have a metastable distribution for each starting profile of the logit dynamics (cf. Section 4). These distributions remain stable for a time which is super-polynomial in $n$, if one is content of being within a distance $\varepsilon > 0$ from the distributions. (The distance is defined in this context as the total variation distance, see below.) To maintain $n$ as our only parameter of interest, we assume that the logarithm of the number of strategies available to players is upper bounded by a polynomial in $n$; this assumption can, however, be relaxed to prove bounds asymptotic in $n$ and in the logarithm of the maximum number of strategies.

The main idea behind our result is that when the dynamics starts from a subset from which it is “hard” to leave, then the dynamics will stay for a long time close to the stationary distribution restricted to that subset. Moreover, if a subset is “easy” to leave, then the dynamics will quickly reach an “hard-to-leave” subset. Our sufficient property has a rather technical definition that is intuitively a classification of subsets that are asymptotically “hard-to-leave” or “easy-to-leave”.

We prove existence of metastable distributions and that the convergence rate, called pseudo-mixing time, is polynomial in $n$ for values of $\beta$ not too big when compared to the (inverse of the) maximum difference in potential of neighboring profiles. Note that when $\beta$ is very high then logit dynamics is “close” to best-response dynamics and therefore it is impossible to prove in general quick convergence results for potential games due to the aforementioned hardness results. We then give a picture which is, in a sense, as complete as possible relatively to the class of potential games for which an asymptotic classification of subsets is possible.

The proofs of the above results build on a number of involved technical contributions, some of which might be of independent interest. They mainly concern Markov chains. The concepts of interest are mixing time (how long the chain takes to mix), bottleneck ratio (intuitively, how hard it is for the stationary distribution to leave a subset of states), hitting time (how long the chain takes to hit a certain subset of states) and spectral properties of the transition matrix of Markov chains.

To describe the metastable distributions of interest, we define a procedure which iteratively identifies in the set of pure strategy profiles the “hard-to-leave” subsets. To prove that the pseudo-mixing time is polynomial in $n$ when the starting profile belongs to the “core” of these distributions, we firstly relate the pseudo-mixing time to the mixing time of a certain family of restricted Markov chains. We then prove that the mixing time of these chains is polynomial by using a spectral characterization of the transition matrix of restricted Markov chains. Finally, the proof that the pseudo-mixing time is polynomial when the dynamics starts outside the “core” mainly relies on a connection between bottleneck ratio and hitting time. Specifically, we prove both an upper bound and a lower bound on the hitting time of a subset of states in terms of the bottleneck ratio of its complement.

Despite the technicality of its definition, we show in Section 5 that our sufficient condition turns out useful for essentially closing an open problem of [4] about metastability of the Curie-Weiss game. This application also shows that, even if the procedure defined above for computing the interested distributions is unpractical, our ideas and technical contributions can be adopted and specialized to obtain a more concrete description of the transient phase of logit dynamics.

We complement the above contributions with further spectral results about the transition matrix of Markov chains defined by logit dynamics for a strategic (not necessarily potential) game (cf. Section 6). These results enhance our understanding of the dynamics and pave the way to further advancements in the area.

**Related works.** Blume [6] introduced logit dynamics for modeling a noisy-rational behavior in game dynamics. Early works about this dynamics have focused on its long-term behavior: Blume [6] showed that, for $2 \times 2$ coordination games and potential games, the long-term behavior of the system is concentrated around a specific Nash equilibrium; Alós-Ferrer and Netzer [1] gave a general characterization of
long-term behavior of logit dynamics for wider classes of games. Several works gave bounds on the time that the dynamics takes to reach specific Nash equilibria of a game: Ellison [13] considered graphical coordination games on cliques and rings; Peyton Young [26] and Montanari and Saberi [23] extended this work to more general families of graphs; Asadpour and Saberi [2] focused on a class of congestion games. Auleta et al. [5] were the first to propose the stationary distribution of the logit dynamics Markov chain as a new equilibrium concept in game theory and to focus on the time the dynamics takes to get close to this equilibrium [3].

In physics, chemistry, and biology, metastability is a phenomenon related to the evolution of systems under noisy dynamics. In particular, metastability concerns moves between regions of the state spaces and the existence of multiple, well separated time scales: at short time scales, the system appears to be in a quasi-equilibrium, but really explores only a confined region of the available space state, while, at larger time scales, it undergoes transitions between such different regions. Research in physics about metastability aims at expressing typical features of a metastable state and to evaluate the transition time between metastable states. Several monographs on the subject are available in physics literature (see, for example, [15, 24, 7, 16]). Auletta et al. [4] applied metastability to probability distributions, introducing the concepts of metastable distribution and pseudo-mixing time for some specific potential games.

Roughly speaking, metastability is a kind of approximation for stationarity. From this point of view, metastable distributions may be likened to approximate equilibria. Two different approaches to approximated equilibria have been proposed in literature. In the multiplicative version [9] a profile is an approximate equilibrium as long as each player gains at least a factor $(1 - \varepsilon)$ of the payoff she gets by playing any other strategy: these equilibria have been shown to be computationally hard both in general [10] and for congestion games [29]. In the additive version [18], a profile is an approximate equilibrium as long as each player gains at least the payoff she gains by playing any other strategy minus a small additive factor $\varepsilon > 0$: for these equilibria a quasi-polynomial time approximation scheme exists [21] but it is impossible to have an FPTAS [8].

2 Preliminary definitions

A strategic game $G$ is a triple $([n], S, \mathcal{U})$, where $[n] = \{1, \ldots, n\}$ is a finite set of players, $S = (S_1, \ldots, S_n)$ is a family of non-empty finite sets ($S_i$ is the set of strategies available to player $i$), and $\mathcal{U} = (u_1, \ldots, u_n)$ is a family of utility functions (or payoffs), where $u_i : S \to \mathbb{R}$, $S = S_1 \times \ldots \times S_n$, being the set of all strategy profiles, is the utility function of player $i$. We let $m$ denote an upper bound to the size of players’ strategy sets, that is, $m \geq \max_{i=1,\ldots,n}|S_i|$. We focus on (exact) potential games, i.e., games for which there exists a function $\Phi : S \to \mathbb{R}$ such that for any pair of $x, y \in S$, $y = (x_{-i}, y_i)$, we have:

$$\Phi(x) - \Phi(y) = u_i(y) - u_i(x).$$

Note that we use the standard game theoretic notation $(x_{-i}, s)$ to mean the vector obtained from $x$ by replacing the $i$-th entry with $s$; i.e. $(x_{-i}, s) = (x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)$. A strategy profile $x$ is a Nash equilibrium\footnote{In this paper, we only focus on pure Nash equilibria. We avoid explicitly mentioning it throughout.} if, for all $i$, $u_i(x) \geq u_i(x_{-i}, s_i)$, for all $s_i \in S_i$. It is fairly easy to see that local minima of the potential function correspond to the Nash equilibria of the game.

For two vectors $x, y$, we denote with $H(x, y) = |\{i : x_i \neq y_i\}|$ the Hamming distance between $x$ and $y$. For every $x \in S$, $N(x) = \{y \in S : H(x, y) = 1\}$ denotes the set of neighbors of $x$ and $N_i(x) = \{y \in N(x) : y_{-i} = x_{-i}\}$ is the set of those neighbors that differ exactly in the $i$-th coordinate.

In this paper, given a set of profiles $L$ we let $\overline{L}$ denote its complementary set, i.e., $\overline{L} = S \setminus L$. 

\[3\]
2.1 Logit dynamics

The logit dynamics has been introduced in [6] and runs as follows: at every time step (i) Select one player \(i\) \(\in\{1, \ldots, n\}\) uniformly at random; (ii) Update the strategy of player \(i\) according to the Boltzmann distribution with parameter \(\beta\) over the set \(S_i\) of her strategies. That is, a strategy \(s_i\) \(\in S_i\) will be selected with probability

\[
\sigma_i(s_i \mid x_{-i}) = \frac{1}{Z_i(x_{-i})} e^{\beta u_i(x_{-i}, s_i)},
\]

where \(x_{-i}\) denotes the profile of strategies played at the current time step by players different from \(i\), 
\(Z_i(x_{-i}) = \sum_{s_i \in S_i} e^{\beta u_i(x_{-i}, s_i)}\) is the normalizing factor, and \(\beta \geq 0\). One can see parameter \(\beta\) as the inverse of the noise or, equivalently, the rationality level of the system: indeed, from (1), it is easy to see that for \(\beta = 0\) player \(i\) selects her strategy uniformly at random, for \(\beta > 0\) the probability is biased toward strategies promising higher payoffs, and for \(\beta\) that goes to infinity player \(i\) chooses her best response strategy (if more than one best response is available, she chooses one of them uniformly at random).

The above dynamics defines a Markov chain \(\{X_t\}_{t \in \mathbb{N}}\) with the set of strategy profiles as state space, and where the transition probability from profile \(x = (x_1, \ldots, x_n)\) to profile \(y = (y_1, \ldots, y_n)\), denoted \(P(x, y) = P_x(X_1 = y)^N\) is zero if \(H(x, y) \geq 2\) and it is \(\frac{1}{n} \sigma_i(y_i \mid x_{-i})\) if the two profiles differ exactly at player \(i\). More formally, we can define the logit dynamics as follows.

**Definition 2.1.** Let \(G = ([n], S, U)\) be a strategic game and let \(\beta \geq 0\). The logit dynamics for \(G\) is the Markov chain \(\mathcal{M}_\beta = ([X_t]_{t \in \mathbb{N}}, S, P)\) where \(S = S_1 \times \cdots \times S_n\) and

\[
P(x, y) = \frac{1}{n} \cdot \begin{cases} 
\sigma_i(y_i \mid x_{-i}), & \text{if } y_{-i} = x_{-i} \text{ and } y_i \neq x_i; \\
\sum_{i=1}^n \sigma_i(y_i \mid x_{-i}), & \text{if } y = x; \\
0, & \text{otherwise}; 
\end{cases}
\]

where \(\sigma_i(y_i \mid x_{-i})\) is defined in (1).

The Markov chain defined by (2) is ergodic [6]. Hence, from every initial profile \(x\) the distribution \(P^t(x, \cdot)\) over states of \(S\) of the chain \(X_t\) starting at \(x\) will eventually converge to a stationary distribution \(\pi\) as \(t\) tends to infinity. As in [5], we call the stationary distribution \(\pi\) of the Markov chain defined by the logit dynamics on a game \(G\), the logit equilibrium of \(G\). In general, a Markov chain with transition matrix \(P\) and state space \(S\) is said to be reversible with respect to a distribution \(\pi\) if, for all \(x, y \in S\), it holds that \(\pi(x) P(x, y) = \pi(y) P(y, x)\). If an ergodic chain is reversible with respect to \(\pi\), then \(\pi\) is its stationary distribution. Therefore when this happens, to simplify our exposition we simply say that the matrix \(P\) is reversible. For the class of potential games the stationary distribution is the well-known Gibbs measure.

**Theorem 2.1** ([6]). If \(G = ([n], S, U)\) is a potential game with potential function \(\Phi\), then the Markov chain given by (2) is reversible with respect to the Gibbs measure \(\pi(x) = \frac{1}{Z} e^{-\beta \Phi(x)}\), where \(Z = \sum_{y \in S} e^{-\beta \Phi(y)}\) is the normalizing constant.

It is worthwhile to notice that logit dynamics for potential games and Glauber dynamics for Gibbs distributions are two ways of looking at the same Markov chain (see [5] for details). This, in particular, implies that we can write

\[
\sigma_i(s_i \mid x_{-i}) = \frac{e^{-\beta \Phi(x_{-i}, s_i)}}{\sum_{z \in S_i} e^{-\beta \Phi(x_{-i}, z)}}.
\]

\(\footnote{Throughout this work, we denote with \(P_x(\cdot)\) the probability of an event conditioned on the starting state of the logit dynamics being \(x\).}^2\)

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4
2.2 Convergence of Markov chains

**Mixing time.** Arguably, the principal notion to measure the rate of convergence of a Markov chain to its stationary distribution is the *mixing time*, which is defined as follows. Let us set

\[ d(t) = \max_{x \in S} \| P^t(x, \cdot) - \pi \|_{TV}, \]

where the *total variation distance* \( \| \mu - \nu \|_{TV} \) between two probability distributions \( \mu \) and \( \nu \) on the same state space \( S \) is defined as

\[ \| \mu - \nu \|_{TV} = \max_{A \subset S} | \mu(A) - \nu(A) | = \frac{1}{2} \sum_{x \in S} | \mu(x) - \nu(x) |. \]

For \( 0 < \varepsilon < 1/2 \), the mixing time of the logit dynamics is defined as

\[ t_{\text{mix}}(\varepsilon) = \min \{ t \in \mathbb{N} : d(t) \leq \varepsilon \}. \]

It is usual to set \( \varepsilon = 1/4 \) or \( \varepsilon = 1/2e \). We write \( t_{\text{mix}} \) to mean \( t_{\text{mix}}(1/4) \) and we refer generically to “mixing time” when the actual value of \( \varepsilon \) is immaterial. Observe that \( t_{\text{mix}}(\varepsilon) \leq \lceil \log_2 \varepsilon^{-1} \rceil t_{\text{mix}} \).

**Relaxation time.** Another important measure of convergence for Markov chains is given by the *relaxation time*. Let \( P \) be the transition matrix of a Markov chain with finite state space \( S \); let us label the eigenvalues of \( P \) in non-increasing order \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|S|} \).

It is well-known (see, for example, Lemma 12.1 in [20]) that \( \lambda_1 = 1 \) and, if \( P \) is ergodic, then \( \lambda_2 < 1 \) and \( \lambda_{|S|} > -1 \). We set \( \lambda^* \) as the largest eigenvalue in absolute value other than \( \lambda_1 \),

\[ \lambda^* = \max_{i=2,\ldots,|S|} \{ |\lambda_i| \} . \]

The relaxation time \( t_{\text{rel}} \) of a Markov chain \( M \) is defined as

\[ t_{\text{rel}} = \frac{1}{1 - \lambda^*}. \]

The relaxation time is related to the mixing time by the following theorem (see, for example, Theorems 12.3 and 12.4 in [20]).

**Theorem 2.2** (Relaxation time). Let \( P \) be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space \( S \) and stationary distribution \( \pi \). Then

\[ (t_{\text{rel}} - 1) \log 2 \leq t_{\text{mix}} \leq \log \left( \frac{4}{\pi_{\min}} \right) t_{\text{rel}}, \]

where \( \pi_{\min} = \min_{x \in S} \pi(x) \).

**Hitting time.** In some cases, we are interested in bounding the first time that the chain hits a profile in a certain set of states, also known as its *hitting time*. Formally, for a set \( L \subseteq S \), we denote by \( \tau_L \) the random variable denoting the hitting time of \( L \). Note that the hitting time, differently from mixing and relaxation time, depends on where the dynamics starts. Some useful fact about hitting time are summarized in Appendix A.
Bottleneck ratio. Quite central in our study is the concept of bottleneck ratio. Consider an ergodic Markov chain with finite state space $S$, transition matrix $P$, and stationary distribution $\pi$. The probability distribution $Q(x, y) = \pi(x)P(x, y)$ is of particular interest and is sometimes called the edge stationary distribution. Note that if the chain is reversible then $Q(x, y) = Q(y, x)$. For any $L \subseteq S$, let $Q(L, S \setminus L) = \sum_{x \in L, y \in S \setminus L} Q(x, y)$. Then the bottleneck ratio of $L$ is

$$B(L) = \frac{Q(L, S \setminus L)}{\pi(L)}.$$  

We use the following theorem to derive lower bounds to the mixing time (see, for example, Theorem 7.3 in [20]).

**Theorem 2.3 (Bottleneck ratio).** Let $\mathcal{M} = \{X_t: t \in \mathbb{N}\}$ be an irreducible and aperiodic Markov chain with finite state space $S$, transition matrix $P$, and stationary distribution $\pi$. Let $L \subseteq S$ be any set with $\pi(L) \leq 1/2$. Then the mixing time is

$$t_{\text{max}} \geq \frac{1}{4B(L)}.$$  

The bottleneck ratio is also strictly related to the relaxation time. Indeed, let

$$B_* = \min_{L: \pi(R) \leq 1/2} B(L),$$

then the following theorem holds (see, for example, Theorem 13.14 in [20]).

**Theorem 2.4.** Let $P$ be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space $S$. Let $\lambda_2$ be the second largest eigenvalue of $P$. Then

$$\frac{B^2}{2} \leq 1 - \lambda_2 \leq 2B_*.$$  

3 Metastability

In this section we give formal definitions of metastable distributions and pseudo-mixing time. We also survey some of the tools used for our results. For a more detailed description we refer the reader to [4].

**Definition 3.1.** Let $P$ be the transition matrix of a Markov chain with finite state space $S$. A probability distribution $\mu$ over $S$ is $(\varepsilon, \mathcal{T})$-metastable for $P$ (or simply metastable, for short) if for every $0 \leq t \leq \mathcal{T}$ it holds that

$$\|\mu P^t - \mu\|_{TV} \leq \varepsilon.$$  

The definition of metastable distribution captures the idea of a distribution that behaves approximately like the stationary distribution: if we start from such a distribution and run the chain we stay close to it for a “long” time. Some interesting properties of metastable distributions are discussed in [4], including the following lemmata, that turn out to be useful for proving our results.

**Lemma 3.1 ([4]).** Let $P$ be a Markov chain with finite state space $S$ and stationary distribution $\pi$. For a subset of states $L \subseteq S$ let $\pi_L$ be the stationary distribution conditioned on $L$, i.e.

$$\pi_L(x) = \begin{cases} \pi(x)/\pi(L), & \text{if } x \in L; \\ 0, & \text{otherwise}. \end{cases} \quad (3)$$

Then, $\pi_L$ is $(B(L), 1)$-metastable.
Lemma 3.2 (4). If \( \mu \) is \((\varepsilon, 1)\)-metatable for \( P \) then \( \mu \) is \((\varepsilon T, T)\)-metatable for \( P \).

Among all metastable distributions, we are interested in the ones that are quickly reached from a (possibly large) set of states. This motivates the following definition.

**Definition 3.2.** Let \( P \) be the transition matrix of a Markov chain with state space \( S \), let \( L \subseteq S \) be a non-empty set of states and let \( \mu \) be a probability distribution over \( S \). We define the *pseudo-mixing time* \( t^L_\mu(\varepsilon) \) as

\[
    t^L_\mu(\varepsilon) = \inf \{ t \in \mathbb{N} : \| P^t(x, \cdot) - \mu \|_{TV} \leq \varepsilon \text{ for all } x \in L \}.
\]

Since the stationary distribution \( \pi \) of an ergodic Markov chain is reached within \( \varepsilon \) in time \( t_{\text{mix}}(\varepsilon) \) from every state, according to Definition 3.2 we have that \( t^S_\pi(\varepsilon) = t_{\text{mix}}(\varepsilon) \). The following simple lemma connects metastability and pseudo-mixing time.

**Lemma 3.3 (4).** Let \( \mu \) be a \((\varepsilon, T)\)-metatable distribution and let \( L \subseteq S \) be a set of states such that \( t^L_\mu(\varepsilon) \) is finite. Then for every \( x \in L \) it holds that \( \| P^t(x, \cdot) - \mu \|_{TV} \leq 2\varepsilon \) for every \( t \leq t^L_\mu(\varepsilon) \).

### 3.1 Asymptotic metastability

The notions and results introduced above apply to a single Markov chain. Auletta et al. 4 adopted these notions to evaluate the behavior of the logit dynamics for potential games, as \( n \) grows. Therefore, we do not have a single Markov chain but a sequence of them, one for each number \( n \) of players, and need to consider an asymptotic counterpart of the notions above. Auletta et al., in fact, showed that the logit dynamics for specific classes of \( n \)-player potential games enjoys the following property, that we name asymptotic metastability.

**Definition 3.3.** Let \( G \) be an \( n \)-player strategic game. We say that the logit dynamics for \( G \) is *asymptotically metastable* for the rationality level \( \beta \) if there are constants \( n_0, \varepsilon > 0 \), a polynomial \( p = p_\varepsilon \) in \( n \) and a super-polynomial \( q = q_\varepsilon \) in \( n \) such that for each \( n \geq n_0 \), the logit dynamics for the \( n \)-player game \( G \) converges in time at most \( p(n) \) from each profile of \( G \) to a \((\varepsilon, q(n))\)-metatable distribution.

Note that we are considering games as an object that can be studied asymptotically in the number of players \( n \). Towards this end, we assume that there is only one instance of the game for each \( n \). This may appear too strong an assumption: e.g., adding one player to a congestion game (known to be a potential game [27]) gives rise to as many different games (i.e., potential functions) as the possible sets of resources the additional player is interested in. However, we observe that our results allow to have this assumption without loss of generality for potential games. We can, in fact, always represent a potential game with multiple instances for a fixed \( n \) as a set of \( n \)-player potential games with a single potential for every \( n \). Roughly speaking, instead of seeing the game “horizontally”, that is as a sequence of sets of instances, where each set contains all instances defined for some specified number of players, we see the game “vertically”, that is as a set of sequences of instances, where each sequence contains at most one instance for each number of players. Since we will prove results that hold for any of these “vertical point-of-view” potential games, then they hold also for any “horizontal point-of-view” game. Unfortunately, next lemma shows that there is a potential game that is not asymptotically metastable.

**Lemma 3.4.** There is a \( n \)-player potential game \( G \) for which the logit dynamics is not asymptotically metastable for any \( \beta \) sufficiently high and any \( \varepsilon < \frac{1}{2} \).

**Proof.** We will show a game \( G \) and a profile \( x \) of this game such that for any \( 0 < \varepsilon < 1/4 \), for infinitely many value of \( n \) and for each polynomial \( p \) in \( n \) and each super-polynomial \( q \) in \( n \), the logit dynamics for \( G \) does not converge in time at most \( p(n) \) from \( x \) to any \((\varepsilon, q(n))\)-metatable distribution, even if the mixing time of the dynamics is larger than \( p \).
Consider the following pairs \((p_j, q_j)\), where \(p_j = n^j\) and \(q_j = \exp\left(\log n \cdot \log^{(j)} n\right)\), where \(\log^{(j)}\) is the functional iteration of the logarithm function. Let us denote as \(n_j\) a value such that \(p_j(n_j) < q_j(n_j) - \varepsilon\). Such a value surely exists since \(p\) is polynomial and \(q\) is super-polynomial. Moreover, observe that for each \(n > n_j\), we have \(p_j(n) < q_j(n) - \varepsilon\). Thus, we can assume without loss of generality that \(1 = n_0 < n_1 < n_2 < \ldots\). Now let \(T\) be a function that is asymptotically sandwiched between \(p_j\) and \(q_j\), for any \(j\). This can be guaranteed by letting \(T\) be a function such that \(T(n) = q_j(n) - \varepsilon\) for \(j\) such that \(n_{j-1} < n \leq n_j\). Note that for any \(p_j\) and for any \(n \geq n_j\), we have \(T(n) = q_k(n) - \varepsilon > p_k(n) \geq p_j(n)\), where \(k \geq j\) is such that \(n_{k-1} < n \leq n_k\). Similarly, for any \(q_j\) and for any \(n \geq n_j\) we have \(T(n) = q_k(n) - \varepsilon < q_k(n) \leq q_j(n)\). (The situation is depicted in Figure 1)

\[
\begin{align*}
T(n) & \leq q_j(n) - \varepsilon \\
& < q_j(n) \\
& = q_k(n) - \varepsilon \\
& \geq p_k(n) \\
& \geq p_j(n)
\end{align*}
\]

Figure 1: For \(n = j, j + 1\) we plot the values of the inverse of the bottleneck ratio of any non-empty subset of profiles with stationary distribution at most \(1/2\). The figure shows how there is a unique subset with super-polynomial \(B^{-1}\) and how \(T(n)\) is built around the functions \(p_j\)’s and \(q_j\)’s.

Let \(G\) be a \(n\)-player game such that each player has exactly two strategies, say 0 and 1. Consider the potential function \(\Phi\) such that for any \(t = 0, \ldots, n - 1\) and any profile \(x\) wherein exactly \(t\) players play strategy 1 we have \(\Phi(x) = n - t\), whereas the profile \(\Phi(1, \ldots, 1) = 1 + k_n, k_n = \frac{1}{\beta} \log \left(\frac{T(n)}{\varepsilon} - 1\right)\).

Observe that if there is a pair \((p, q)\) with \(p\) polynomial in \(n\) and \(q\) super-polynomial in \(n\) such that it is possible to prove that the logit dynamics for \(G\) is asymptotically metastable with parameters \(p\) and \(q\), then there is \(j^*\) such that the results holds also with \((p_{j^*}, q_{j^*})\) in place of \(p\) and \(q\). Hence, in order to prove the lemma is sufficient to show that it holds only for pairs \((p_j, q_j)\) as described above.

Note that, by taking \(\beta\) sufficiently high, we have that: (i) \(\pi(0, \ldots, 0) \geq \frac{1}{2}\); (ii) there exists a \(j\) such that, for any subset \(L \subseteq \{0, 1\}^n \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}\), the bottleneck ratio \(B(L)\) is at least the inverse of \(p_j\); (iii) the bottleneck ratio \(B(1, \ldots, 1) = \frac{\varepsilon}{T(n)}\).

Firstly note that the mixing time of logit dynamics for \(G\) is not polynomial. Indeed, from Theorem 2.3 it follows that the mixing time is at least \(\frac{T(n)}{\varepsilon}\). However, as suggested above, for each \(p_j\), we have \(\frac{\varepsilon}{T(n)} > T > p_j\) for infinitely many \(n\), and hence the mixing time is asymptotically greater than any polynomial \(p_j\).

We next discuss that no metastable distribution is stable for a superpolynomial time or, even if there is one, it cannot be reached in polynomial time. From Lemma 5.1 and Lemma 5.2 we have that for each \(n\) the distribution \(\pi_1\) that assigns probability 1 to the profile \((1, \ldots, 1)\) is \((\varepsilon, T(n))\)-metastable. However, as suggested above, for each \(q_j\), the function \(T\) is smaller than \(q_j\) for infinitely many \(n\). Thus, the distribution \(\pi_1\) is metastable for time that is asymptotically smaller than any super-polynomial \(q_j\).
Note that this argument extends to any \((3\epsilon, T(n))\)-metastable distribution \(\mu\) that is within distance \(2\epsilon\) from \(\pi_1\). Finally, observe that, the remaining distributions that are far from \(\pi_1\) cannot be reached quickly from \((1, \ldots, 1)\). In fact, from Lemma 4.11 for any polynomial \(p_j\) the probability that the logit dynamics leaves the profile \((1, \ldots, 1)\) in \(p_j\) steps, is at most \(\frac{\epsilon}{p_j} < \epsilon\), for \(n\) sufficiently large. Hence, for any \(p_j\), starting from \((1, \ldots, 1)\) the pseudo-mixing time of any distribution \(\mu\) that is at least \(2\epsilon\)-far from \(\pi_1\) is asymptotically greater than \(p_j\).

\(\square\)

Remark 3.1. The game described in the proof of Lemma 3.4 also shows the necessity of having a definition of asymptotic metastability, as the one given in Definition 3.3.

Consider, indeed, the weaker definition of asymptotic metastability in which for each \(n\) there is a polynomial \(p_n(n)\) and a super-polynomial \(q_n(n)\) governing convergence and stability time of metastability, respectively (i.e., a definition in which the order of quantifiers is reversed). This concept might at first glance look meaningful. However, it is instead of scarce significance as the distinction between polynomials and super-polynomials might become null in the limit.

The game in Lemma 3.4 exemplifies this phenomenon, since it does not satisfies the metastability notion given in the one in Definition 3.3, but it satisfies this weak notion. Indeed, for any \(n\), there is a super-polynomial function, namely \(q_n = q_j - \epsilon\) for \(j\) such that \(n_{j-1} < n \leq n_j\), such that \(\pi_1\) is \((\epsilon, q_n(n))\)-metastable. Obviously, the pseudo-mixing time of this distribution from the profile \((1, \ldots, 1)\) is 1. From the remaining profiles, the dynamics quickly converges to the stationary distribution for any \(\beta\) sufficiently large (this follows from well-known results about birth-and-death chains).

Since a general result on potential games is not possible, the aim of next section is then to devise conditions under which it is possible to prove asymptotic metastability of the logit dynamics for potential games.

4. A sufficient condition for metastability

In this section we will show a condition on \(n\)-player potential games sufficient to prove that for each starting profile \(x\) there is a distribution \(\mu\) metastable for super-polynomial time and whose pseudo-mixing time from \(x\) is polynomial.

Henceforth, we will assume that the logarithm of the maximal number of strategies available to a player is at most a polynomial in \(n\). Specifically, we denote as \(m(\cdot)\) the function such that \(m(n)\) is the maximum number of strategies available to a player in \(G\) when the number of players is \(n\). Then, we will assume that the function \(\log m(\cdot)\) is at most polynomial in its input. We can easily drop this assumption by asking for results that are asymptotic in \(\log |S|\), where \(|S|\) denotes the function returning the number of profiles of the game: each one of our proof can be rewritten according to this requirement with very small changes. Note that having results asymptotic in the logarithm of the number of states is a common requirement in Markov chain literature. Moreover, since \(|S|\) for a game with \(n\) players is at most \(m(n)^n\), this requirement is equivalent to asking for results asymptotic in \(n\) and in the logarithm of the function \(m\).

Note also that we focus only on \(n\)-player potential games and values of \(\beta\) such that the mixing time of the logit dynamics is at least super-polynomial in \(n\), otherwise the stationary distribution enjoys the desired properties of stability and convergence. Throughout this section we will denote with \(\beta_0\) the smaller value of \(\beta\) such that the mixing time is not polynomial.

4.1 Asymptotically well-defined games

To get an intuition of our sufficient condition it is worth looking more closely at the proof of Lemma 3.4. The game of Lemma 3.4 necessitates the update of the function \(T\) infinitely often when adding a new player. This update is done so that the new profile \((1, \ldots, 1)\) has a bottleneck ratio that cannot
be described by any of the functions considered at that point. This process is never ending and gives
no asymptotic meaning to $T$ in the limit. The intuition is then that a game cannot be asymptotically
metastable when for each choice of a polynomial $p$, a super-polynomial $q$ and infinitely many values of
$n$, there is a subset of states $L$ with $n$ players such that $\pi(L) < 1/2$ and

$$p(n) < B^{-1}(L) < q(n).$$

The l.h.s. roughly implies that we cannot leave $L$ within $p$. This in turns must mean that there is a
metastable distribution with support within $L$; a candidate is, from Lemma 3.1 $\pi_{L'}$, $L' \subseteq L$. However,
the r.h.s. roughly implies that this distribution will not be stable for a time $q$. For such a game, we cannot
therefore find two functions $p$ and $q$ as requested by Definition 3.3.

Our sufficient property assumes indeed that we can asymptotically classify the bottleneck ratio of
each subset of profiles as either polynomial or super-polynomial. More specifically, we assume there are
two functions $p$ at most polynomial in the input and $q$ at least super-polynomial in the input such that for
each $\beta$ and each subset $A$ the bottleneck ratio $B(A)$ can be bounded by functions that depends on either
$p$ or $q$. (The kind of functions will be clarified in the next sections.) An equivalent viewpoint would
be to see $n$-player potential games as a class to which a kind of oracle is attached that distinguishes
between polynomial and super-polynomial bottleneck ratios for any fixed $\beta$. Formally, given a $n$-player
potential game $G$ and fixed $\beta \geq \beta_0$, this oracle can be described as follows: when it is queried about the
bottleneck ratio of a subset $A$ with $n$ players its answer states that the bottleneck ratio is either i) at most
polynomial if it is lower-bounded by $1/p(n)$; or ii) at least super-polynomial if it is upper-bounded by
$1/q(n)$. We call an $n$-player potential game enjoying this property asymptotically well-defined (AWD).

4.2 Main result and proof idea

We denote with $\Delta(\cdot)$ the function that, for every $n$, gives the Lipschitz constant of the potential function
$\Phi$ for a game $G$ with $n$ players, i.e.,

$$\Delta(n) := \max \{ \Phi(x) - \Phi(y) : H(x, y) = 1 \}.$$

The main result of this paper is stated below.

**Theorem 4.1.** Let $G$ be an AWD $n$-player potential game and let $\Delta(n)$ as defined above. Fix constant
$\varepsilon > 0$ and a function $\rho$ at most polynomial in its input. Then, the logit dynamics for $G$ is asymptotically
metastable for each $\beta_0 \leq \beta \leq \rho(n)/\Delta(n)$.

A high level idea of the proof is discussed next. We initially need to define the metastable distributions
to which the logit dynamics converges. To describe the distributions of interest for the given
$n$-player potential game $G$, we leverage known results connecting mixing time, bottleneck ratio and
metastability. In particular, it turns out that super-polynomial mixing time implies the existence of a
subset of profiles with bottleneck ratio at most the inverse of a super-polynomial (see Theorem 2.2 and
Theorem 2.4). Then, the stationary distribution of the dynamics restricted to this subset, as defined in
(3), turns out to be metastable for a super-polynomial amount of time (Theorem 3.1). In this way we can
easily “build” a metastable distribution whenever the mixing time of the dynamics is high (cf. Lemma
4.1).

What about the pseudo-mixing time of this distribution? We distinguish two cases. First we consider
profiles that are in the “core” of the support of this distribution. We show that the pseudo-mixing time
from these profiles is related to the mixing time of a special restriction of the original dynamics (see
Corollary 4.1). To compute the mixing time of these restricted dynamics, we give a spectral character-
ization of their transition matrix (Proposition 4.2) and use again the relationship among mixing time,
relaxation time and bottleneck ratio (see Theorems 2.2 and 2.4).
It may be the case that the mixing time of the restricted dynamics is not polynomial. As pointed above, to high mixing time there corresponds a subset with small bottleneck ratio. This suggests that we can use this smaller subset of profiles as a support of a new metastable distribution and compute the pseudo-mixing time of this distribution as described above. The problem is that the bottleneck ratio of a subset depends on the dynamics according to which it is computed. Thus, a subset can have a small bottleneck ratio when computed within the reference frame of the restricted dynamics, but not when we refer to the original dynamics. Nevertheless, we show that this is not the case (Lemma 4.4). Specifically, we will show that there is a relationship between the bottleneck ratio of a subset of profiles in the restricted and in the original dynamics. Thus, it is possible to iteratively consider smaller candidates as metastable distributions as long as we find one whose pseudo-mixing time from the core is polynomial. For sake of presentation, we simply show that, whenever we are given the smallest subset with bottleneck ratio at most the inverse of a super-polynomial, the pseudo-mixing time of the corresponding distribution from the core is polynomial.

What about out-of-core profiles? Suppose that there is a profile from which the dynamics takes long time to converge to a metastable distribution. Then it must be the case that the dynamics takes long time to hit the core of one such distribution with high probability. We then show that there is a strong relationship between hitting time and metastability (see Lemma 4.11 and Lemma 4.12) and, in particular, that high hitting time implies the existence of a subset with small bottleneck ratio and, thus, of a new metastable distribution. Hence, once that all metastable distributions have been found, it must be the case that the dynamics quickly converges to one of these distributions, and thus to a still metastable convex distribution of them (Lemma 4.2), from each remaining out-of-core profile.

Proof presentation. The proof starts by describing the metastable distributions in Section 4.3. We then bound the pseudo-mixing time when the starting profile is in the core of a metastable distribution (Section 4.4) and when the starting profile is out of the “core” (Section 4.5). Our analysis for the pseudo-mixing time assumes that \( \beta \) (Section 4.4) and the starting profile is out of the “core” (Section 4.5).

For every piece of our proof, we introduce the necessary technical tools first: in particular, the main tools adopted in our proofs are represented by Proposition 4.2, which gives a spectral characterization for suitable restriction of the logit dynamics transition matrix, Corollary 4.1 that relates the pseudo-mixing time to the mixing time of the restricted chains described in (4), and by Lemma 4.11 and Lemma 4.12 that, instead, relate the hitting time to the bottleneck ratio.

4.3 Metastable distributions

Let \( \Phi \) be a potential function on profile space \( S \). Let \( P \) be the transition matrix of the logit dynamics on \( \Phi \) and let \( \pi \) be the corresponding stationary distribution. For \( L \subseteq S \) non-empty, we define a Markov chain with state space \( L \) and transition matrix \( \hat{P}_L \) defined as follows.

\[
P_L(x, y) = \begin{cases} 
P(x, y) & \text{if } x \neq y; \\ 
1 - \sum_{z \in L} P(x, z) = P(x, x) + \sum_{z \in S \setminus L} P(x, z) & \text{otherwise.} 
\end{cases}
\] (4)

It easy to check that the stationary distribution of this Markov chain is given by the distribution \( \pi_L(x) = \frac{\pi(x)}{\pi(L)} \), for every \( x \in L \). Note also that the Markov chain defined upon \( \hat{P}_L \) is reversible and aperiodic, since the Markov chain defined upon \( P \) is, and it will be irreducible if \( L \) is a connected set. For a fixed \( \varepsilon > 0 \), we will denote with \( t_{\text{mix}}^L(\varepsilon) \) the mixing time of the chain described in (4). We also denote with \( B_L(A) \) the bottleneck ratio of \( A \subset L \) in the Markov chain with state space \( L \) and transition matrix \( P_L \).

We now introduce an algorithm that defines some subsets \( R_1, \ldots, R_k \) of the set \( S \) of strategy profiles, with \( k \geq 1 \). Moreover, this algorithm partitions \( S \) in \( k + 1 \) subsets \( T_1, \ldots, T_k \) and \( N \), where \( T_1, \ldots, T_k \)
represent the core of the sets $R_1, \ldots, R_k$ and the last subset $N$ simply contains the remaining profiles of $S$. The procedure works its way by finding subsets of profiles that act as super-polynomial bottlenecks for the Markov chain. The algorithm $A_{p,q}$ is parametrized by two functions $p$ at most polynomial and $q$ at least super-polynomial. It takes in input an $n$-player potential game $G$, a rationality level $\beta$, a constant $\varepsilon > 0$ and $n$.

**Algorithm 4.1 ($A_{p,q}$).** Set $N = S$ and $i = 1$. While there is a set $L \subseteq N$ with $\pi(L) \leq 1/2$ such that $B(L) \leq 1/q(n)$, do:

1. Denote with $R_i$ one such subset with the smallest stationary probability;
2. Denote with $T_i$ the largest subset of $R_i$ such that for every $y \in T_i$,
   \[ P_y \left( \tau_{S \setminus R_i} \leq \frac{R_i}{\max(\varepsilon)} \right) \leq \varepsilon; \]
3. If $T_i$ is not empty, return $R_i$ and $T_i$, delete from $N$ all profiles contained in $T_i$ and increase $i$. Otherwise, terminate the algorithm.

Observe that if there is a disconnected set $L$ such that $B(L) \leq 1/q(n)$, then each connected component $L'$ of $L$ will have $B(L') \leq 1/q(n)$ and smaller stationary probability: hence, the set $R_i$ returned by the algorithm will be connected. Note also that by Theorem 2.3 and the assumption of super-polynomial mixing time, the algorithm above enters at least once in the loop (and thus at least a subset $R_i$ is computed).

The following lemmata prove that certain distributions defined on the sets $R_i$ returned by Algorithm 4.1 are metastable for super-polynomial time.

**Lemma 4.1.** Let $G$ be an AWD $n$-player potential game and consider the stationary distribution $\pi$ of the logit dynamics for $G$. Fix $\beta \geq \beta_0$. There exists a function $T = T_{R_i}$ at least super-polynomial in the input such that for each $i$, and $n$ large enough, the distribution $\mu_i$ that sets $\mu_i(x) = \pi(x)/\pi(R_i)$ is $(\varepsilon, T(n))$-metastable for every $\varepsilon > 0$.

**Proof.** Fix $i$. Given $\varepsilon > 0$, consider the function $T = T_{R_i}$ such that $T(n) = \frac{\varepsilon}{\pi(R_i)} \geq \varepsilon q(n)$, where $R_i$ is the support of $\mu_i$. By the definition of $q$, $T$ is at least super-polynomial in the input.

By Lemma 3.1, $\mu_i$ is $(B(R_i), 1)$-metastable. By Lemma 3.2, $\mu_i$ is also $(B(R_i) \cdot T(n), T(n))$-metastable. The lemma follows since $B(R_i) \cdot T(n) = \varepsilon$. $\blacksquare$

Finally, the following lemma shows that a combination of metastable distributions is metastable.

**Lemma 4.2.** Let $P$ the transition matrix of a Markov chain with state space $S$ and let $\mu$ be a distribution $(\varepsilon, T_i)$-metastable for $P$, for $i = 1, 2, \ldots$. Set $\varepsilon = \max \{ \varepsilon_i \}$ and $T = \max \{ T_i \}$. Then, the distribution $\mu = \sum_i \alpha_i \mu_i$, with $\sum_i \alpha_i = 1$ and $\alpha_i \geq 0$, is $(\varepsilon, T)$-metastable.

**Proof.** For every $t \leq T$ we have

\[ ||\mu^t - \mu||_{TV} = \max_{A \subseteq S} \left| \left( \mu^t(A) - \mu(A) \right) \right| \]

\[ = \max_{A \subseteq S} \sum_i \alpha_i \left| \left( \mu_i^t(A) - \mu_i(A) \right) \right| \]

\[ \leq \sum_i \alpha_i \max_{A \subseteq S} \left| \left( \mu_i^t(A) - \mu_i(A) \right) \right| \leq \varepsilon. \]

Note: There can be values of $n$ for which the algorithm does not run the $i$-th iteration and thus $R_i$, $T_i$, and $\mu_i$ are not well defined. However, as long as there are infinite values of $n$ for which $R_i$ is computed then asymptotic bounds on the pseudo-mixing time of $\mu_i$ from $T_i$ are well defined. Since the algorithm executes at least one iteration for any input, we have that there exists $\max_{n \geq n_0} k(n)$, $R_i$ is computed infinite times.
4.4 Pseudo-mixing time starting from \( T_i \)

In this section, we will prove that the logit dynamics for an AWD \( n \)-player potential game and \( \beta \) small enough converges in polynomial time to the metastable distribution \( \mu_i \) defined above, whenever the starting point is selected from the core \( T_i \) of this distribution. Specifically, we prove the following proposition.

**Proposition 4.1.** Let \( G \) be an AWD \( n \)-player potential game; fix \( \varepsilon > 0 \) and a function \( \rho \) at most polynomial in its input. There exists a function \( p_\star \) at most polynomial in the input such that for each \( i \), \( n \) large enough and \( \beta_0 \leq \beta \leq \frac{\rho(n)}{\Delta(n)} \), the pseudo-mixing time of \( \mu_i \) from \( T_i \) is \( t_{\mu_i}^{R_i}(2\varepsilon) = O(p_\star(n)) \).

We first prove a useful spectral property enjoyed by the restriction (4) of the original dynamics. This property will turn out to be useful for proving in Section 4.4.2 that the mixing time \( t_{\text{mix}}^{R_i}(\varepsilon) \) of this restricted dynamics is polynomial. Finally, in Section 4.4.3 we show that the previous result is sufficient to prove the proposition.

### 4.4.1 Spectral property of logit dynamics restrictions

In [3] it has been shown that all the eigenvalues of the transition matrix of logit dynamics for potential games are non-negative. The technique used in that proof can be generalized to work also for some restrictions of these matrices.

To begin, we note that the definition of reversibility can be extended in a natural way to any square matrix and probability distribution over the set of rows of the matrix. We then state a fairly standard result relating eigenvalues of matrices to certain inner products.

**Lemma 4.3.** Let \( P \) be a square matrix on state space \( S \) and \( \pi \) be a probability distribution on \( S \). If \( P \) is reversible with respect to \( \pi \) and has no negative eigenvalues then for any function \( f : S \to \mathbb{R} \) we have

\[
\langle Pf,f \rangle_\pi := \sum_{x \in S} \pi(x)(Pf)(x)f(x) \geq 0.
\]

**Proof.** Let \( \lambda_1, \ldots, \lambda_s, s = |S|, \) be the eigenvalues of \( P \). Moreover, let \( f_1, \ldots, f_s \) denote their corresponding eigenfunctions. For any \( x \in S \), we then have \( (Pf_i)(x)f_i(x) = \lambda_if_i(x) \). Since \( P \) is reversible then we know that the eigenfunctions assume real values and that they form an orthonormal basis for the space \( (\mathbb{R}^s, \langle \cdot, \cdot \rangle_\pi) \) (see, e.g., Lemma 12.2 in [20]). Then any real-valued function \( f \) defined upon \( S \) can be expressed as a linear combination of the \( f_i \)'s. Thus, there exist \( \alpha_i \)'s in \( \mathbb{R} \) such that

\[
\sum_{x \in S} \pi(x)(Pf)(x)f(x) = \sum_{x \in S} \pi(x) \sum_{i=1}^s \alpha_i^2(Pf_i)(x)f_i(x) = \sum_{x \in S} \pi(x) \sum_{i=1}^s \alpha_i^2 \lambda_i f_i^2 (x) \geq 0. \quad \Box
\]

To specify the restrictions of the transition matrix we are interested in, let \( G \) be a game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( G \); we say that a \([A] \times [A]\) matrix \( P' \), with \( A \subseteq S \), is a nice restriction of \( P \) if there exists \( L \subseteq A \), \( L \not= \emptyset \), such that \( P'(x,y) \geq P(x,y) \) for \( x \in L \), \( P'(x,y) = P(x,y) \) if \( x,y \in L \), \( x \not= y \), and is 0 otherwise. Note that \( P \) is a nice restriction of itself. We generalize the result given in [3] to nice restrictions of the transition matrix of logit dynamics for potential games.

**Proposition 4.2.** Let \( G \) be a game with profile space \( S \), let \( P \) be the transition matrix of the logit dynamics for \( G \) and let \( P' \) be a nice restriction of \( P \) with state space \( A \). If \( P \) is reversible then no eigenvalue of \( P' \) is negative.
Proof. Firstly, note that if \( P \) is reversible with respect to \( \pi \) then the nice restriction \( P' \), defined upon a subset of states \( A \), is reversible with respect to \( \pi' \) defined as \( \pi' \) restricted to \( A \), i.e., \( \pi'(x) = \pi(x)/\pi(A) \) for \( x \in A \).

Assume for sake of contradiction that there exists an eigenvalue \( \lambda < 0 \) of \( P' \). Let \( f_\lambda \) be an eigen-function of \( \lambda \). Note that since \( P \) is reversible then \( f_\lambda \) is real-valued. By definition, \( f_\lambda \neq 0 \); hence, since \( \lambda < 0 \) and as \( (P'f_\lambda)(x) = \lambda f_\lambda(x) \), then for every profile \( x \in A \) such that \( f_\lambda(x) \neq 0 \) we have \( \text{sign} ((P'f_\lambda)(x)) \neq \text{sign} (f_\lambda(x)) \) and thus

\[
\langle P'f_\lambda, f_\lambda \rangle_{\pi'} = \sum_{x \in A} \pi'(x)(P'f_\lambda)(x)f_\lambda(x) < 0.
\]

Let \( L \) denote the maximal subset of \( A \) for which \( P' \) is a nice restriction of \( P \). Let us denote with \( P^L \) the transition matrix on the state space \( A \) such that \( P^L(x,y) = P(x,y) \) for every \( x, y \in L \) and \( P^L(x,y) = 0 \) otherwise. Then we can write \( P' \) as \( P^L + (P' - P^L) \); by the definition of nice restriction \( (P' - P^L) \) is a non-negative diagonal matrix. Therefore, \( (P' - P^L) \) is reversible with respect to \( \pi' \).

Since the eigenvalues of a diagonal matrix are exactly the diagonal elements, we have that \( (P' - P^L) \) has non-negative eigenvalues and then, by Lemma 4.3, \( \langle (P' - P^L)f_\lambda, f_\lambda \rangle_{\pi'} \geq 0 \). Moreover, for every \( i \) and for every \( z_{-i} \), we denote with \( P_{i,z_{-i}} \) the matrix such that for every \( x, y \in A \)

\[
P_{i,z_{-i}}(x,y) = \begin{cases} 
\frac{1}{nZ_i(z_{-i})} e^{\beta\hat{u}_i(y)}, & \text{if } x_{-i} = y_{-i} = z_{-i} \text{ and } x, y \in L; \\
0, & \text{otherwise.}
\end{cases}
\]

Observe that \( P_{i,z_{-i}} \) has at least one non-zero row and that all non-zero rows of \( P_{i,z_{-i}} \) are the same. Thus \( P_{i,z_{-i}} \) has rank 1, and hence since it is a non-negative matrix all its eigenvalues are non-negative. Moreover, since all off-diagonal entries of \( P_{i,z_{-i}} \) are either 0 or equal to the corresponding entry of \( P' \) we can conclude that \( P_{i,z_{-i}} \) is reversible with respect to \( \pi' \). Thus, Lemma 4.3 yields \( \langle P_{i,z_{-i}}f_\lambda, f_\lambda \rangle_{\pi'} \geq 0 \). Finally, observe that \( P^L = \sum_i \sum_{z_{-i}} P_{i,z_{-i}} \). Hence from the linearity of the inner product, it follows that \( \langle P'f_\lambda, f_\lambda \rangle_{\pi'} \geq 0 \) and thus we reach a contradiction. \( \square \)

It is immediate to see that the restricted chain \( \hat{P}_L \) defined in (4) is a nice restriction of \( P \) and hence all its eigenvalues are non-negative by the theorem above. This will turn out to be very useful to prove Proposition 4.1.

4.4.2 Mixing time of the restricted chain

Before to bound the mixing time of the restricted chain we prove a very important preliminary lemma.

Lemma 4.4. Let \( G \) be an AWD \( n \)-player potential game and fix \( \beta \geq \beta_0, \varepsilon > 0 \). Let \( p, q \) the functions for which \( G \) is AWD. Consider the sequence of sets \( R_i \) returned by \( A_{p,q} \). Then, for \( n \) sufficiently large, we have

\[
B_{R_i}(A) \geq \frac{1}{p(n)} - \frac{1}{\ell(n)},
\]

where \( \ell \) is at least super-polynomial.

Proof. Let us postpone the exact definition of \( \ell \) and suppose, by contradiction, that there are infinitely many \( n \) for which there is \( A \subset R_i \) such that \( B_{R_i}(A) < \frac{1}{p(n)} - \frac{1}{\ell(n)} \).

We will show that for \( n \) sufficiently large either \( B(A) \leq 1/q(n) \) or \( B(\overline{A}) \leq 1/q(n) \). Then, since they are contained in \( R_i \) and hence their stationary probability is less than \( \pi(R_i) \), one of these set must be chosen before \( R_i \) by \( A_{p,q} \). But since in the third step of Algorithm 4.1 either at least one element
of such sets should be deleted from $N$ or the algorithm terminates, as a consequence, we have that $R_i$ cannot be returned by the algorithm, thus a contradiction.

Consider the function $v(\cdot)$ that sets $v(n) = \frac{\pi(A)_{\pi(A,S \setminus R_i)}}{Q(A,S \setminus R_i)}$. We distinguish two cases depending on how $v$ evolves as $n$ grows.

If $v(\cdot)$ is at least super-polynomial in the input: We have

$$B(A) = \frac{Q(A,S \setminus A)}{\pi(A)} = \frac{Q(A,R_i \setminus A)}{\pi(A)} + \frac{Q(A,S \setminus R_i)}{\pi(A)}$$

$$= \sum_{x \in A} \sum_{y \in R_i \setminus A} \pi(x)P(x,y) + \frac{Q(A,S \setminus R_i)}{\pi(A)}$$

$$= \sum_{x \in A} \sum_{y \in R_i \setminus A} \pi_{R_i}(x)P_{R_i}(x,y) + \frac{Q(A,S \setminus R_i)}{\pi(A)}$$

$$= B_{R_i}(A) + \frac{Q(A,S \setminus R_i)}{\pi(A)} < \frac{1}{p(n)} + \frac{1}{v(n)} - \frac{1}{\ell(n)}.$$

By taking $\ell(n) \leq v(n)$ for each $n$ sufficiently large, we have that $B(A) < \frac{1}{p(n)}$. Hence, since $G$ is AWD, $B(A) \leq \frac{1}{q(n)}$.

If $v(\cdot)$ is polynomial in the input: Note that $\frac{Q(A,S \setminus R_i)}{\pi(R_i)} + \frac{Q(A,S \setminus R_i)}{\pi(R_i)} = B(R_i) \leq \frac{1}{q(n)}$, otherwise $R_i$ was not returned by the algorithm. Hence, we obtain

$$Q(A,S \setminus R_i) \leq \frac{1}{q(n)} \cdot \pi(R_i) \quad \text{and} \quad Q(A,S \setminus R_i) \leq \frac{1}{q(n)} \cdot \pi(R_i).$$

From the first of these inequalities, we have $\pi(A) \leq \frac{v(n)}{q(n)} \cdot \pi(R_i)$. Hence

$$\frac{Q(A,A)}{\pi(R_i)} \leq \frac{v(n)}{q(n)} \cdot \frac{Q(A,A)}{\pi(A)} = \frac{v(n)}{q(n)} \cdot B_{R_i}(A) < \frac{v(n)}{q(n)} \left( \frac{1}{p(n)} - \frac{1}{\ell(n)} \right).$$

Then we obtain

$$B(A) = \frac{Q(A,S \setminus A)}{\pi(A)} = \frac{Q(A,A)}{\pi(R_i) - \pi(A)} + \frac{Q(A,S \setminus R_i)}{\pi(R_i) - \pi(A)}$$

(by reversibility of $P$)

$$\leq \frac{v(n)}{q(n)^2} \left( \frac{1}{p(n)} - \frac{1}{\ell(n)} \right) \left( 1 - \frac{v(n)}{q(n)} \right)^{-1} + \frac{1}{q(n)} \left( 1 - \frac{v(n)}{q(n)} \right)^{-1}$$

$$= O \left( \frac{1}{q(n)} \right),$$

where the upper bounds hold for each choice of super-polynomial function $\ell$. Since $q(n) - v(n)$ evolves at least as a super-polynomial, if $n$ is sufficiently large, $B(A) < \frac{1}{p(n)}$. Hence, since $G$ is AWD, $B(A) \leq \frac{1}{q(n)}$.

Now we are ready to prove the mixing time of the chain restricted to $R_i$ is polynomial.

**Lemma 4.5.** Let $G$ be an AWD $n$-player potential game; fix $\varepsilon > 0$ and a function $\rho$ at most polynomial in its input. Let $p, q$ the functions for which $G$ is AWD. Consider the sequence of sets $R_i$ returned by $A_{p,q}$. For any $n$ sufficiently large, if $\beta_0 \leq \beta \leq \frac{\rho(n)}{\Delta(n)}$ then $\tau_{mix}(\varepsilon)$ is at most polynomial.
Proof. Consider the set of profiles $A_\ast \subset R_i$ that minimizes $B_{R_i}(A)$ among all $A \subset R_i$ such that $\pi_{R_i}(A) \leq 1/2$. By Lemma 4.4, $B_{R_i}(A_\ast) \geq 1/p(n) - 1/\ell(n)$ for each $n$ sufficiently large.

Moreover, for each $n$ and each $x \in R_i$, since $|S| \leq m(n)^n$, it follows that

$$\log \frac{1}{\pi_{R_i}(x)} \leq |S| e^{-\beta \Phi_{\min}} \leq \log \frac{e^n \log m(n) e^{-\beta \Phi_{\min}}}{e^{-\beta \Phi_{\max}}} = n \log m(n) + \beta (\Phi_{\max} - \Phi_{\min}),$$

where $\Phi_{\max}$ and $\Phi_{\min}$ denote the maximum and minimum of the potential $\Phi$ overall possible strategy profiles. Since $\Phi_{\max} - \Phi_{\min} \leq n \cdot \Delta(n)$ and $\beta \leq \rho(n)/\Delta(n)$, then

$$\log \frac{1}{\pi_{R_i}(x)} \leq n \cdot \log m(n) + \rho(n).$$

Then, since $\left(\frac{1}{p} - \frac{1}{\ell}\right) = \Theta\left(\frac{1}{p}\right)$, from Proposition 4.2 and the properties of the relaxation time (see Theorems 2.4 and 2.2) it follows that the mixing time is

$$\tau_{R_i}^{\text{mix}}(1/4) \leq \left(\frac{1}{p(n)} - \frac{1}{\ell(n)}\right)^{-2} \cdot (n \log m(n) + \rho(n)) \cdot 2 \log \frac{4}{\varepsilon} = O(p_\ast(n)).$$

Since $p$, $\log m$ and $\rho$ are at most polynomial, then $p_\ast$ is at most polynomial in its input and the lemma follows. \qed

4.4.3 Pseudo-mixing time

For $L \subseteq S$ non-empty, consider the Markov chain defined in (4). Let us abuse the notation and denote with $P_L$ and $\pi_L$ also the Markov chain and the distribution defined on the entire state space $S$, assuming $P_L(x, y) = 0$ if $x \not\in L$ or $y \not\in L$, and similarly $\pi_L(x) = 0$ when $x \not\in L$: reversibility and non-negativity of eigenvalues continue to hold also in this case.

For $L \subseteq S$ we set $\partial L$ as the border of $L$, that is the set of profiles in $L$ with at least a neighbor in $S \setminus L$. Recall that $\tau_{S \setminus L}$ is the random variable denoting the first time the Markov chain with transition matrix $P$ hits a profile $x \in S \setminus L$. The following lemma formally proves the intuitive fact that, by starting from a profile in $L$ the chain $P$ and the chain $\hat{P}_L$ are the same up to the time in which the former chain hits a profile in $S \setminus L$. The proof uses the well-known coupling technique (cf., e.g., [20]) which is summarized in Appendix B.

Lemma 4.6. Let $P$ be the transition matrix of a Markov chain with state space $S$ and let $\hat{P}_L$ be the restriction of $P$ to $L \subseteq S$, $L \neq \emptyset$, as given in (4). Then, for every $x \in L$ and for every $t > 0$,

$$\left\| P^t(x, \cdot) - \hat{P}_L^t(x, \cdot) \right\|_{TV} \leq P_x (\tau_{S \setminus L} \leq t).$$

Proof. Consider the following coupling $(X_t, Y_t)_{t>0}$ of the Markov chains with transition matrix $P$ and $\hat{P}_L$, respectively:

- If $X_t = Y_t \in L \setminus \partial L$, then we update the first chain according to $P$ and obtain $X_{t+1}$; we then set $Y_{t+1} = X_{t+1}$;
- If $X_t = Y_t \in \partial L$, then we update the first chain according to $P$: if $X_{t+1} \in L$, then we set $Y_{t+1} = X_{t+1}$, otherwise we set $Y_{t+1} = Y_t$;
- If $X_t \neq Y_t$, then we update the chains independently.
Since $X_0 = Y_0 = x \in L$, we have that $X_t \neq Y_t$ only if $\tau_{S \setminus L} \leq t$. Thus, by the properties of couplings (see Theorem B.1), we have

$$\left\| P^t(x, \cdot) - \hat{P}_L^t(x, \cdot) \right\|_{TV} \leq P_x(X_t \neq Y_t) \leq P_x(\tau_{S \setminus L} \leq t).$$

The following corollary follows from the Lemma 4.6 and the triangle inequality property of the total variation distance.

**Corollary 4.1.** Let $P$ the transition matrix of a Markov chain with state space $S$ and let $\hat{P}_L$ be the restriction of $P$ to a non-empty $L \subseteq S$ as given in (4). Then, for every $x \in L$ and for every $t > 0$,

$$\left\| P^t(x, \cdot) - \pi_L \right\|_{TV} \leq \left\| \hat{P}_L^t(x, \cdot) - \pi_L \right\|_{TV} + P_x(\tau_{S \setminus L} \leq t).$$

Using Corollary 4.1 we can prove Proposition 4.1.

**Proof of Proposition 4.1.** For each $x \in T_i$, by Corollary 4.1 and since $P_x(\tau_{S \setminus R_i} \leq t^{R_i}(\varepsilon)) \leq \varepsilon$, we obtain

$$\left\| P^{t^{R_i}(\varepsilon)}(x, \cdot) - \mu_i \right\|_{TV} \leq \varepsilon + \varepsilon.$$

The lemma follows from the observation that Lemma 4.5 yields that $t^{R_i}(\varepsilon)$ is at most a polynomial.

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### 4.5 Pseudo-mixing time starting from any profile

In order to bound the pseudo-mixing time from any remaining profile and $\beta$ not too large we first introduce some useful technical tools.

#### 4.5.1 The relation between bottleneck ratio and hitting time

For a game $G$ with potential function $\Phi$ and profile space $S$, and a rationality level $\beta$, let $P$ be the transition matrix of the Markov chain defined by the logit dynamics on $G$. For a non-empty $L \subseteq S$, we denote with $P_L$ the matrix

$$P_L(x, y) = \begin{cases} P(x, y) & \text{if } x, y \in L; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1^L \geq \lambda_2^L \geq \ldots \geq \lambda_{|S|}^L$ be the eigenvalues of $P_L$; notice that $\lambda_i^L$ can be different from 1 since the matrix $P_L$ is not stochastic. Proposition 4.2 implies that $\lambda_1^L \geq \lambda_2^L \geq \ldots \geq \lambda_{|S|}^L \geq 0$, and thus for $\lambda_{\max}$, the largest eigenvalue of $P_L$ in absolute value, we have: $\lambda_{\max}^L = \max_{i \leq |S|} |\lambda_i^L| = \lambda^L_{\max}$.

We start with two characterizations in terms of bottleneck ratio of $1 - \lambda_{\max}^L$. The first one is an easy extension of the similar characterization of the spectral gap of stochastic matrices.

**Lemma 4.7.** For finite $\beta$ and any $\emptyset \neq L \subseteq S$, $1 - \lambda_{\max}^L \leq B(L)$.

**Proof.** Define the function $\varphi_L : S \to [0, 1]$ to be such that $\varphi_L(x) = \pi(L)$ if $x \in L$, and $\varphi_L(x) = 0$ otherwise. Consider now the function

$$E_P(\varphi_L) := \frac{1}{2} \sum_{x, y \in S} \pi(x) P(x, y) (\varphi_L(x) - \varphi_L(y))^2.$$  

(7)
By Theorem 2.1, \( \pi(L) \neq 0 \) and then \( E_\pi[\phi_L^2] = \pi(L)^2 \neq 0 \). Moreover, by denoting with \( \partial L \) the set of profiles \( x \in L \) that have at least one neighbor profile in \( S \setminus L \) and with \( E(A_1, A_2) \) the pairs of neighbor profiles \( (x, y) \) such that \( x \in A_1 \) and \( y \in A_2 \). We have:

\[
E_P(\phi_L) = \frac{\pi(L)}{2}\left( \sum_{(x,y) \in E(L,L)} \pi(x)P(x,y) + \sum_{(x,y) \in E(S \setminus L,L)} \pi(x)P(x,y) \right)
\]

\[
= \pi(L)^2 \sum_{x \in \partial L} \pi(x) \sum_{y \in S \setminus L : H(x,y) = 1} P(x,y) = \pi(L)^2 Q(L, S \setminus L),
\]

where we used the reversibility of \( P \) in the penultimate equality. Hence, we have \( \frac{E_P(\phi_L)}{E_\pi[\phi_L^2]} = B(L) \). The lemma follows since \( 1 - \lambda_{\text{max}}^T \leq \frac{E_P(\phi_L)}{E_\pi[\phi_L^2]} \) (see Lemma A.1 in Appendix). 

The second characterization may be proved in exactly the same way as a similar well-known characterization for the spectral gap of stochastic matrices (see, for example, Section 13.3.3 in [20]).

**Lemma 4.8.** For any \( \emptyset \neq L \subseteq S \),

\[
1 - \lambda_{\text{max}}^T \geq \frac{(B_L)^2}{2}.
\]

Finally, let us recall a couple of lemmata relating \( \tau_{S \setminus L} \) and \( \lambda_{\text{max}}^T \) and already stated in e.g. [23].

**Lemma 4.9.** For a reversible Markov chain with state space \( S \), any \( L \subseteq S \) and any \( t \) it holds that

\[
\max_{x \in L} P_x(\tau_{S \setminus L} > t) \geq \exp \left( t \log \lambda_{\text{max}}^T \right).
\]

**Lemma 4.10.** For a reversible Markov chain with state space \( S \), any \( L \subseteq S \) and any \( t \) it holds that

\[
P_x(\tau_{S \setminus L} > t) \leq \exp \left( t \log \lambda_{\text{max}}^T + \frac{1}{2} \log \frac{1}{\pi_L(x)} \right),
\]

where \( \pi_L(x) \) has been defined in (3).

Since the statement of Lemma 4.10 is slightly different from the ones found in previous literature, we attach a proof in Appendix A for sake of completeness.

The above lemmata represent the main ingredients to prove the following relations between bottleneck ratio and hitting time.

**Lemma 4.11.** Let \( G \) be a potential game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( G \). Then for finite \( \beta \) and \( L \subset S, L \neq \emptyset \), we have

\[
\min_{x \in L} P_x(\tau_{S \setminus L} \leq t) \leq t 
\frac{B(L)}{1 - B(L)}.
\]

**Proof.** We observe:

\[
\min_{x \in L} P_x(\tau_{S \setminus L} \leq t) = 1 - \max_{x \in L} P_x(\tau_L > t)
\]  
(by Lemma 4.9)

\[
\leq 1 - \exp \left( t \log \lambda_{\text{max}}^T \right)
\]

\[
= 1 - \exp \left( t \log (1 - (1 - \lambda_{\text{max}}^T)) \right).
\]
Thus, by setting \( t \) where \( \frac{1}{\lambda_{\max}} \) (by Lemma 4.7)
\[
1 - e^{-a} \leq 1 - \exp \left( -t \cdot \frac{B(L)}{1 - B(L)} \right).
\]

(since \( 1 - a \geq e^{-\frac{a}{1-a}} \)) \leq 1 - \exp \left( -t \cdot \frac{B(L)}{1 - B(L)} \right).

Moreover, for \( x \in L \) and \( 0 < \varepsilon < 1 \), we also let \( T_{S \backslash L}^\varepsilon(x) \) be the first time step \( t \) in which \( P_x(t_{S \backslash L} > t) \leq \varepsilon \). Then we have the following lemma.

**Lemma 4.12.** Let \( G \) be a potential game with profile space \( S \) and \( P \) be the transition matrix of the logit dynamics for \( G \). For \( \beta > 0 \), \( \emptyset \neq L \subseteq S \), \( x \in L \) and \( 0 < \varepsilon < 1 \), let \( T_{S \backslash L}^\varepsilon(x) \) be defined as above (with respect to \( P \)). Then
\[
T_{S \backslash L}^\varepsilon(x) \leq (B_L^M)^{-2} \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_L(x)} \right),
\]
where \( \pi_L(x) = \frac{\pi(x)}{\pi_L(x)} \) and \( B_L^M = \min_{A \subseteq L: \pi(A) \leq 1/2} B(A) \).

**Proof.** From Lemma 4.10 we know that the hitting time of \( S \backslash L \) can be expressed as a function of the eigenvalues of the matrix \( P_L^M \) (see Theorem ). In particular, we have
\[
P_x(t_{S \backslash L} > t) \leq \exp \left( t \log \lambda_{\max}^L + \frac{1}{2} \log \frac{1}{\pi_L(x)} \right).
\]

(since \( 1 - a \leq e^{-a} \)) \leq \exp \left( -t \left( 1 - \lambda_{\max}^L \right) + \frac{1}{2} \log \frac{1}{\pi_L(x)} \right).

(by Lemma 4.8) \leq \exp \left[ \frac{1}{2} \left( t(B_L^M)^2 - \log \frac{1}{\pi_L(x)} \right) \right].

(since \( e^{-a} \leq (1 + a)^{-1} \)) \leq \left( 1 + \frac{1}{2} \left( t(B_L^M)^2 - \log \frac{1}{\pi_L(x)} \right) \right)^{-1}.

Thus, by setting \( t = (B_L^M)^{-2} \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_L(x)} \right) \), we have \( P_x(t_{S \backslash L} > t) \leq \varepsilon \) and then \( T_{S \backslash L}^\varepsilon(x) \) is upper bounded by this value of \( t \).

**4.5.2 Pseudo-mixing time**

Consider the distributions \( \mu_i \) defined above (i.e., the stationary distribution restricted to \( R_i \)). We focus here on the convergence time to distributions of the form
\[
\nu(y) = \sum_i \alpha_i \mu_i(y),
\]
for \( \alpha_i \geq 0 \) and \( \sum_i \alpha_i = 1 \). Specifically, for every profile \( x \in N \), we define the distribution
\[
\nu_x(y) = \sum_i \mu_i(y) \cdot P_x \left( X_{t_{S \backslash N}} \in T_i \mid t_{S \backslash N} \leq T_{S \backslash N}^\varepsilon(x) \right).
\]

Observe that by definition of \( t_{S \backslash N} \), since the \( T_i \)'s and \( N \) are a partition of \( S \), \( X_{t_{S \backslash N}} \in \bigcup_i T_i \) is a certain event for all values of \( t_{S \backslash N} \). Moreover, we show below that we can condition on the event \( t_{S \backslash N} \leq T_{S \backslash N}^\varepsilon(x) \). Thus, \( \sum_i P_x \left( X_{t_{S \backslash N}} \in T_i \mid t_{S \backslash N} \leq T_{S \backslash N}^\varepsilon(x) \right) = 1 \). The above is then a valid definition of the \( \alpha_i \)'s.

Then, we prove the following proposition.
Proposition 4.3. Let $\mathcal{G}$ be an AWD $n$-player potential game; fix $\varepsilon > 0$ and a function $\rho$ at most polynomial in its input. There is a function $T = T_\varepsilon$ at least super-polynomial in the input and a function $p_\ast$ at most polynomial in the input such that for every $n$ large enough if $\beta_0 \leq \beta \leq \frac{\mu(n)}{2^{\log n}}$, then for each $\mathbf{x} \in N$ the corresponding distribution $\nu_\mathbf{x}$ is $(\varepsilon, T(n))$-metastable and the pseudo-mixing time of $\nu_\mathbf{x}$ from the profile $\mathbf{x}$ is $t_{\nu_\mathbf{x}}(3\varepsilon) = O(p_\ast(n))$.

The following lemma turns out to be useful for proving fast convergence.

Lemma 4.13. Let $\mathcal{G}$ be an AWD $n$-player potential game and fix $\beta \geq \beta_0, \varepsilon > 0$. Let $p, q$ the functions for which $\mathcal{G}$ is AWD. Then, for each $n$ sufficiently large, at the end of algorithm $A_{p,q}$ on input $\mathcal{G}, \beta, \varepsilon$ and $n$ it holds that for each subset $L \subseteq N$ such that $\pi(L) \leq 1/2$, $B(L) \geq 1/p(n)$.

Proof. It is sufficient to prove that for each $R_i$ chosen by $A_{p,q}$, its core $T_i$ is non-empty. Indeed, in this case, the algorithm ends only if no subset $L \subseteq N$ such that $\pi(L) \leq 1/2$ has $B(L) > 1/q(n)$. Since $\mathcal{G}$ is AWD, then the last condition is equivalent to $B(L) \geq 1/p(n)$.

As for the non-emptiness of the core, Lemma [4.11] implies that there exists at least one $\mathbf{x} \in R_i$ such that

$$P_\mathbf{x} \left( \tau_{S \setminus R_i} \leq t_{\text{mix}}^{R_i} (\varepsilon) \right) \leq \frac{t_{\text{mix}}^{R_i} (\varepsilon) \cdot B(R_i)}{1 - B(R_i)} \leq \varepsilon,$$

where the last step holds for $n$ sufficiently large since $t_{\text{mix}}^{R_i}$ is at most polynomial by Lemma [4.5] and $B(R_i)$ is at most the inverse of a super-polynomial by hypothesis.

We are now ready to prove Proposition 4.3.

Proof of Proposition 4.3. Notice that, the distribution $\nu_\mathbf{x}$ is a convex combination of distributions that are metastable for super-polynomial time: thus, from Lemma [4.2] there exist a function $T$ at least super-polynomial in the input such that each such $\nu_\mathbf{x}$ is $(\varepsilon, T(n))$-metastable.

Consider now the function $\pi_\mathbf{x}$ for which

$$\pi(\mathbf{x}) = \pi_\mathbf{x}(\mathbf{x}) \leq \frac{1}{B(A_\ast)} \left( 2(1 - \varepsilon) + \log \frac{1}{\pi_N(\mathbf{x})} \right) = p(n)^2 \cdot \left( 2(1 - \varepsilon) + \rho'(n) \right) = \rho_\ast(n),$$

where, by assumption on $m$ and $\rho$, $\rho'$ is a function at most polynomial in its input. Then, for every $\mathbf{x} \in N$, from Lemma [4.12] it follows

$$T_{\mathcal{S} \setminus N}(\mathbf{x}) \leq \left( \frac{1}{B(A_\ast)} \right)^2 \cdot \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_N(\mathbf{x})} \right) \leq p(n)^2 \cdot \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \rho'(n) \right) = \rho_\ast(n),$$

where $\rho_\ast$ is a function at most polynomial in its input, since $p$ and $\rho'$ are.

Consider now the function $p_\ast(\cdot)$ such that $p_\ast(n) = \rho_\ast(n) + \max_i t_{\mu_i}^{R_i}(\varepsilon)$. From Proposition 4.1 and the fact that $\rho_\ast$ is at most polynomial, it turns out that $p_\ast(\cdot)$ is at most a polynomial function in the input.

We complete the proof by showing that, for any sufficiently large $n$ and any $\mathbf{x} \in N$, $p_\ast(n)$ upper bounds the pseudo-mixing time $t_{\nu_\mathbf{x}}(3\varepsilon)$ to $\nu_\mathbf{x}$ from the profile $\mathbf{x}$.
We denote $t^* = p^*(n)$, with $E$ the event “$\tau_{S_i \setminus N} \leq \mathcal{T}_{S_i \setminus N}^\varepsilon(x)$” and with $E$ its complement. Recall from Definition 2.1 that $X_t$ denotes the state of the Markov chain defined by logit dynamics at step $t$ and observe that

$$\|P^{t^*}(x, \cdot) - \nu_x\|_{TV} = \max_{A \subset S} |P_x(X_{t^*} \in A) - \nu_x(A)|$$

$$= \max_{A \subset S} |P_x(X_{t^*} \in A \cap E) - \nu_x(A) + P_x(X_{t^*} \in A \cap \bar{E})|$$

$$= \max_{A \subset S} |P_x(X_{t^*} \in A \mid E)(1 - P_x(\bar{E})) - \nu_x(A) + P_x(X_{t^*} \in A \mid \bar{E}) P_x(\bar{E})|$$

$$\leq \max_{A \subset S} |P_x(X_{t^*} \in A \mid E) - \nu_x(A)| + P_x(\bar{E})$$

$$\leq \|P_x(X_{t^*} \mid E) - \nu_x\|_{TV} + \varepsilon,$$

where the definition of $\mathcal{T}_{S_i \setminus N}^\varepsilon(x)$ implies that $P_x(E) \geq 1 - \varepsilon > 0$ and then yields the third equality and last inequality. The penultimate inequality, instead, simply follows from the subadditivity of the absolute value and the fact that the difference between two probabilities is upper bounded by 1. As every $\mu_i$ is metastable for at least a super-polynomial number of steps, we have, by using $\tau^*$ as a shorthand for $\tau_{S_i \setminus N}$,

$$\|P_x(X_{t^*} \mid E) - \nu_x\|_{TV} = \sum_i \sum_{y \in T_i} P_x(X_{t^*} = y \mid E) \cdot P_x(X_{t^*} \mid X_{t^*} = y \mid E) - \nu_x$$

$$\leq \sum_i \sum_{y \in T_i} P_x(X_{t^*} = y \mid E) \left(P^{t^* - \tau^*}(y, \cdot) - \mu_i\right)$$

$$\leq \sum_i \sum_{y \in T_i} P_x(X_{t^*} = y \mid E) \left(P^{t^* - \tau^*}(y, \cdot) - \mu_i\right)_{TV} \leq 2\varepsilon,$$

where the definition of $\tau^*$ yields $X_{t^*} \in T_i$, for some $i$, which in turns yields the first equality by the law of total probability. In the first inequality above, instead, we use the definition of $\nu_x$ and the fact that by definition of $t^*$, $E$ implies $t^* - \tau^* \geq t^* - \mathcal{T}_{S_i \setminus N}^\varepsilon(x) \geq \max_i \mathcal{T}_{A_i}^\varepsilon(\varepsilon)$; the second inequality follows from a simple union bound; and the last inequality follows from Lemma 3.3 (note that $t^* - \tau^*$ satisfies the hypothesis of the lemma: the lower bound is showed above, while the upper bound follows from the fact that the $\mu_i$’s are metastable for at least super-polynomial time). Hence, we have for every sufficiently large $n$ and every $x \in N$, $\ell_{\beta}(x, (3\varepsilon) \leq t^* = p^*(n)$.}

### 4.6 Asymptotically well-defined games revisited

Technically to prove asymptotic metastability of a game with respect to a polynomial $p$ and a superpolynomial $q$ we do not need the behavior of each subset to be classified as in the definition of asymptotically well-defined games described above. That is, we can allow some subsets of profiles to have bottleneck ratio in between the inverse of $q$ and the inverse of $p$. In this case, we will say that the subset of profiles is unclassified. We now describe which class of subsets is sufficient to classify in order to prove our main result. In particular, we can redefine the class of asymptotically well-defined games as follows.

**Definition 4.1 (Asymptotically well-defined games).** An $n$-player potential games $\mathcal{G}$ is asymptotically well-defined (AWD) if for every $\beta \geq \beta_0, \varepsilon > 0$ there exist a pair of functions $p$ at most polynomial and $q$ at least super-polynomial, that for each $n$ sufficiently large, satisfy the following conditions:

1. $q(n) \leq \max_{L: \pi(L) \leq 1/2} B^{-1}(L)$;

2. for each $R_i$ computed by $A_{p,q}$ and for any $L \subset R_i$ such that $\pi_{R_i}(L) \leq 1/2$, if $B_{R_i}(L) < 1/p(n)$, then both $B(L)$ and $B(R_i \setminus L)$ are not unclassified;
3. for each subset \( L \subseteq N, N \) being as at the end of the algorithm \( A_{p,q} \), such that \( \pi(L) \leq 1/2, B(L) \) is not unclassified.

Then, it is easy to see that Theorem 4.1 continues to hold. Indeed, the AWD property is used twice during the proof above, namely, in the proof of Lemma 4.4 and in the proof of Lemma 4.13. In particular, in Lemma 4.4 in order to achieve the contradiction, we need that for a subset \( A \subset R \) such that \( \pi_{R}(A) \leq 1/2 \) and \( B_{R}(A) < 1/p(n) \) both \( B(A) \) and \( B(A) \) are classified. Clearly, the second requirement in Definition 4.1 assures that this property always holds. As for Lemma 4.13 we need that for a subset \( L \subseteq N, if B(L) > 1/q(n), then B(L) \geq 1/p(n) \). This condition is then satisfied by the third requirement in Definition 4.1.

5 An application: the Curie-Weiss model

We will show in this section how, despite the technicality of the definition of AWD games, the sufficient condition found in the previous section can be used to solve a problem left open in [4].

Consider the following game-theoretic formulation of the well-studied Curie-Weiss model (the Ising model on the complete graph), that we will call CW-game: each one of \( n \) players has two strategies, \(-1\) and \(+1\), and the utility of player \( i \) at profile \( x = (x_1, \ldots, x_n) \in \{-1, +1\}^n \) is \( u_i(x) = x_i \sum_{j \neq i} x_j \).

Observe that for every player \( i \) it holds that

\[
 u_i(x_{-i}, +1) - u_i(x_{-i}, -1) = \mathcal{H}(x_{-i}, -1) - \mathcal{H}(x_{-i}, +1),
\]

where \( \mathcal{H}(x) = -\sum_{j \neq k} x_j x_k \), hence the CW-game is a potential game with potential function \( \mathcal{H} \). The magnetization of \( x \) is defined as \( M(x) = \sum_i x_i \).

It is known (see Chapter 15 in [20]) that the logit dynamics for this game (or equivalently the Glauber dynamics for the Curie-Weiss model) has mixing time polynomial in \( n \) for \( \beta < 1/n \) and super-polynomial as long as \( \beta > 1/n \). Moreover, [4] describes metastable distributions for \( \beta > c \log n/n \) and shows that such distributions are quickly reached from profiles where the number of \(+1\) (respectively \(-1\)) is a sufficiently large majority, namely if the magnetization \( k \) is such that \( k^2 > c \log n/\beta. \) Thus it is left open what happens when \( \beta \) lies in the interval \((1/n, c \log n/n)\) and if a metastable distribution is quickly reached when in the starting point the number of \(+1\) is close to the number of \(-1\). We essentially close this problem by showing that CW-games are AWD for \( \beta \geq c/n \) for some constant \( c > 1 \).

**Theorem 5.1.** Let \( G \) be the \( n \)-player CW-game. Then for any \( \beta \geq c/n \), for constant \( c \geq 1 \) the logit dynamics for \( G \) is asymptotically metastable.

**Proof.** It will be sufficient to show that \( G \) is an AWD game as from Definition 4.1. First of all, note that a super-polynomial function \( q \) exists such that \( q \) lower bounds the maximum bottleneck ratio, as required by the first condition in Definition 4.1. Indeed, as observed, the mixing time of the logit dynamics for \( G \) is super-polynomial for \( \beta \geq c/n \).

Let \( S_+ \) (resp., \( S_- \)) be the set of profiles with positive (resp., negative) magnetization and \( \pi_+ \) (resp., \( \pi_- \)) be the restriction of the stationary distribution to \( S_+ \) (resp., \( S_- \)). We first observe that the mixing time of the chain restricted to \( S_+ \) (resp. \( S_- \)) is polynomial. In particular, in [19] it has been proved that this mixing time is actually \( c_1 n \log n \) for some constant \( c_1 > 0 \).

Let now \( \zeta \) be the unique positive root of the function

\[
 f(x) = \frac{e^{\beta x} (1 - x) - e^{-\beta x} (1 + x)}{e^{\beta x} (1 - x) + e^{-\beta x} (1 + x)}. \]

\[ \text{The result in [19] refers to censored chains, that are exactly the same as our restricted chain, except that the probability that the original chain from a profile \( x \) goes out from \( L \) is "reflected" to some profile in \( L \) different from \( x \), instead than to be "added" to the probability to do not leave \( x \). It is immediate to see how their result extends also to our restricted chains.} \]
Let \( Z \) be the set of profiles with magnetization \( k \) such that \(|k| \geq \zeta n\). We prove that from any from any profile \( \mathbf{x} \in Z \) the dynamics hits a profile \( \mathbf{y} \in S_+ \) in polynomial time with probability at most \( \varepsilon \). Consider, indeed, the magnetization chain, i.e., the birth and death chain on the space \( \{-n, 2 - n, \ldots, n - 2, n\} \). Then we are interested in the hitting time \( \tau_l \) of \( l \leq 0 \) when the starting point is \( k \) (or, symmetrically, the hitting time of \( l \geq 0 \) when the starting point is \(-k\)). Clearly, in order to reach magnetization \( l \) it is necessary to reach magnetization \( k' \), with \( l < k' < k \). And for reaching \( l \) from \( k' \) it is necessary to reach \( k'' \) such that \( l \leq k'' < k' \). Then, we show that there is \( k' \) from which the chain quickly goes back to \( n\zeta \) with high probability without ever hitting the profile \( k'' \). In particular, in \cite[Theorem 4.10]{11} it has been showed that there are \( k' \) and \( k'' \) such that
\[
\Pr_{k'} \left( \tau_{n\zeta} \leq c_2 n \log n \land \tau_{k''} \geq c_2 n \log n \right) \geq 1 - o(1).
\]
Hence, it easily follows that
\[
\Pr_k \left( \tau_l \geq c_1 n \log n \right) \geq (1 - o(1))^{c_1/c_2} = 1 - o(1).
\]

Consider now, the set \( \mathcal{Z} = S \setminus Z \) of remaining profiles. From \cite[Theorems 4.4, 4.9 and 4.10]{11} (see also \cite{12}), we have that for each profile \( \mathbf{x} \in \mathcal{Z} \), the hitting time of \( Z \) is polynomial with high probability. From Lemma \cite[4.1]{41} and Lemma \cite[4.12]{42} it then follows that for each subset of \( Z \) the bottleneck ratio is polynomial.

Hence, since the sets \( R_i \) returned by Algorithm \cite[4.1]{41} have super-polynomial bottleneck ratio, then each \( R_i \) must contain a profile \( \mathbf{x} \in Z \). Also, it must be the case that \( \pi_{R_i} \) is close to either \( \pi_+ \) or \( \pi_- \). Otherwise, let \( \mathbf{y} \in S_+ \) be a profile that has high probability in \( \pi_+ \) but not in \( \pi_{R_i} \). Then, since the mixing time of the Markov chain restricted to \( S_+ \) is polynomial and that starting from \( \mathbf{x} \) the chain stays in \( S_+ \) with high probability it follows that \( \mathbf{y} \) is visited with probability very close to \( \pi_+(y) \) in polynomial time. But since \( \pi_+(y) \) is larger than \( \pi_{R_i}(y) \), then it will be the case that the dynamics is far from \( \pi_{R_i} \) after polynomial time, contradicting that the bottleneck ratio of \( R_i \) is super-polynomial.

Now, since the mixing time of \( \pi_+ \) and \( \pi_- \) is polynomial, the mixing time of the dynamics restricted to \( R_i \) is also polynomial. In other words, there exists a polynomial \( p \) such that no subset \( L \subset R_i \) with \( \pi_{R_i}(L) \leq \frac{1}{2} \) and \( B_{R_i}(L) < 1/p(n) \) exists. Thus, the second condition of Definition \cite[4.1]{41} is satisfied.

Finally, from what we discussed above, it follows that any profile \( \mathbf{x} \in \mathcal{Z} \) either belongs to the core of some metastable distribution or from it the dynamics quickly hits a profile in the core of a metastable distribution. Similarly, it turns out that for every profile in \( A \) the hitting time of a metastable distribution is polynomial with high probability. Then, the third condition in Definition \cite[4.1]{41} is also satisfied, completing the proof of the theorem.

For the Curie-Weiss game, and in general for any AWD game, the metastable distributions can be described through the sets returned by Algorithm \cite[4.1]{41} However, the algorithm is unpractical and does not allow to explicitly define the metastable distributions. Hence, since we know that such distributions exist it is natural to ask how we can find a more explicit description of metastable distributions for specific games. The proof of Theorem \cite[5.1]{51} allows us to give such an explicit description for the Curie-Weiss game.

**Corollary 5.1.** Let \( \mathcal{G} \) be the \( n \)-player CW-game and consider the logit dynamics for \( \mathcal{G} \). If \( \beta \geq c/n \), with constant \( c > 1 \), then \( \pi_+ \) and \( \pi_- \), as defined above, are \( (\varepsilon, T) \)-metastable, with \( \varepsilon > 0 \) and \( T \) exponential in \( n \). Moreover, for every starting profile the logit dynamics reaches a convex combination of these distributions in polynomial time.

### 6 Spectral properties of the logit dynamics

We next give other interesting spectral results about the transition matrix generated by the logit dynamics. In particular, by using a matrix decomposition similar to the one adopted in the proof of Proposi-
exactly the sum of the traces of all dynamics for \( G \). The trace of \( P \) is independent of \( \beta \).

Proof. For every \( i \) and for every \( z_{-i} \) consider the transition matrices \( P_{i,z_{-i}} \) defined in (5), with \( L = S \). Let \( S_{i,z_{-i}} = \{ (z_{-i}, s_i) \mid s_i \in S_i \} \). Observe that for every \( x \in S_{i,z_{-i}} \) we have \( P_{i,z_{-i}}(x, x) = 1 - \sum_{y \in S_{i,z_{-i}}, y \neq x} P(x, y) \). Hence, the trace of \( P_{i,z_{-i}} \) is

\[
\sum_{x \in S_{i,z_{-i}}} P_{i,z_{-i}}(x, x) = |S_i| - \sum_{x \in S_{i,z_{-i}}} \sum_{y \in S_{i,z_{-i}}, y \neq x} P(x, y).
\]

Since all non-zero elements in a column of \( P_{i,z_{-i}} \) are the same we also have

\[
P_{i,z_{-i}}(x, x) = \frac{1}{|S_i| - 1} \sum_{y \in S_{i,z_{-i}}, y \neq x} P(y, x).
\]

By setting \( C = \sum_{x \in S_{i,z_{-i}}} \sum_{y \in S_{i,z_{-i}}, y \neq x} P(x, y) = \sum_{x \in S_{i,z_{-i}}} \sum_{y \in S_{i,z_{-i}}, y \neq x} P(y, x) \), we have

\[
|S_i| - C = \frac{C}{|S_i| - 1} \implies C = |S_i| - 1,
\]

and thus, the trace of \( P_{i,z_{-i}} \) is always 1, regardless of \( \beta \). The proposition follows since the trace of \( P \) is exactly the sum of the traces of all \( P_{i,z_{-i}} \)'s.

The proposition above says that if there exists an eigenvalue of \( P \) that gets closer to 1 as \( \beta \) increases, then there are other eigenvalues that get smaller: this is very promising in the tentative to characterize the entire spectrum of eigenvalues of \( P \), necessary to use powerful tools such as the well-known random target lemma [20].

In order to prove our last characterization of the transition matrix generated by the logit dynamics, we prove the following lemma which gives a lower bound on the probability that the strategy profile is not changed in one step of the logit dynamics for a generic game.

Lemma 6.1. Let \( G \) be a game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( G \). Then for every \( x \in S \) we have that

\[
P(x, x) = \sum_i P((x_{-i}, s_i^*), x),
\]

where \( s_i^* \neq x_i \) is an arbitrary strategy of player \( i \).

Proof. Observe that

\[
P(x, x) = 1 - \sum_{y \in N(x)} P(x, y) = \sum_i \left( \frac{1}{n} - \sum_{y \in N_i(x)} P(x, y) \right) = \sum_i \left( 1 - \sum_{y \in N_i(x)} e^{\beta u_i(y)} + \sum_{z \in N_i(x)} e^{\beta u_i(z)} \right) = \sum_i \left( \frac{1}{n} e^{\beta u_i(x)} + \sum_{z \in N_i(x)} e^{\beta u_i(z)} \right).
\]

The proof concludes by observing that for every \( i \) and for every \( s_i^* \in S_i \), we have

\[
P((x_{-i}, s_i^*), x) = \frac{1}{n} e^{\beta u_i(x)} + \sum_{z \in N_i(x)} e^{\beta u_i(z)}. \]

24
Lemma 6.1 allows us to calculate the determinant of $P$.

**Proposition 6.2.** Let $G$ be a game with profile space $S$ and let $P$ be the transition matrix of the logit dynamics for $G$. The determinant of $P$ is 0.

**Proof.** It is well-known that a matrix in which one row can be expressed as a linear combination of other rows has determinant zero. In this proof, we fix a profile $x$ and show that the row of $P$ corresponding to $x$ can be obtained as a linear combination of other rows of the matrix. For each player $i$, fix a strategy $s_i^* \in S_i$ such that $s_i^* \neq x_i$. Let us denote with $S_j^i$, $j = 0, \ldots, n$, the set of profiles $y \in S$ obtained from $x$ by selecting $j$ players $i_1, \ldots, i_j$ and setting their strategies to $s_{i_1}^*, \ldots, s_{i_j}^*$, respectively. Notice that $x$ belongs to $S^0$. By construction, for every profile $z \in S^1$, $z_i \in \{x_i, s_i^*\}$. Now, for $i = 1, \ldots, n$, consider the profile obtained from $z$ by changing $z_i = x_i$ into $s_i^*$ or vice versa. Note that there are $n$ such profiles which are neighbors of $z$ and all contained in the sets $S_j^{i-1}$ and $S_j^{i+1}$. We claim that for every $y \in S$

$$P(x, y) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{y \in S_j^i} P(z, y). \quad (9)$$

In order to prove the claim we distinguish three cases:

1. Let $H(x, y) > 1$ (and thus $P(x, y) = 0$): if there exists $j \in \{0, \ldots, n\}$ such that $y \in S_j^i$, then the r.h.s. of (9) becomes $\pm \left( P(y, y) - \sum_{i} P((y_{i-1}, s_i^*), y) \right) = 0$, from Lemma 6.1 if $y \notin \bigcup_{j=0}^{n} S_j^i$. Then consider a profile $z \in S^1$, for some $j = 1, \ldots, n$, such that $z$ differs from $y$ only in the strategy of player $k$: if no such profile exists, then the r.h.s. of (9) is 0; otherwise, let us assume w.l.o.g. $z_k = x_k$ (the case $z_k = s_k^*$ can be managed similarly), then the profile $z' = (z_{-k}, s_k^*)$ is a neighbor of $y$, belongs to the set $S_j^{i+1}$ and $P(z, y) = P(z', y)$: hence, this two profiles delete each other in the r.h.s. of (9), giving the aimed result.

2. Let $x, y$ differ in the strategy adopted by the player $k$: if there exists $j \in \{0, \ldots, n\}$ such that $y \in S_j^i$, then the r.h.s. of (9) becomes $P(y, y) - \sum_{i \neq k} P((y_{i-1}, s_i^*), y) = P(x, y)$, from Lemma 6.1 if $y \notin \bigcup_{j=0}^{n} S_j^i$, then, as above, all profiles in $\bigcup_{j=0}^{n} S_j^i$ that differ from $y$ only in one player $i \neq k$ delete each other in the r.h.s. of (9): thus, the only element that survives in the r.h.s. of (9) is $P((x_{-k}, x_k), y) = P(x, y)$.

3. If $x = y$, then the r.h.s. of (9) becomes $\sum_{i \neq k} P((y_{i-1}, s_i^*), y) = P(x, x)$, from Lemma 6.1. $\square$

Since, as observed above, logit dynamics for potential games defines a reversible Markov chain, Proposition 4.2 and Proposition 6.2 imply that the last eigenvalue of the logit dynamics for these games is exactly 0. (Note that in [3] is only stated the last eigenvalue is non-negative.) Moreover, from the proof above, it turns out that an eigenvector of such zero eigenvalue is given by the function $f: S \to \mathbb{R}$ defined as

$$f(w) = \begin{cases} 
-1, & \text{if } w \in S_j^i \text{ and } j \text{ is even}; \\
1, & \text{if } w \in S_j^i \text{ and } j \text{ is odd}; \\
0, & \text{otherwise}; 
\end{cases}$$

where the sets $S_j^i$’s are defined as in the above proof from some fixed profile $x$.

### 7 Conclusions and open problems

In this work we study sufficient conditions for the metastability of logit dynamics for potential games. Our property is *game-independent* and related to the asymptotic behavior of certain distributions. It is not clear, and a very interesting open problem, what property is instead *necessary* in order to have
metastable distributions for potential games. It seems that in order to answer this question, new ideas are needed (either new distributions have to be considered or new mathematical tools used) or game-specific arguments ought to be used (the proofs given in [4] suggest that these arguments can be very involved also for very simple games).

Our convergence rate results hold if $\beta$ is small enough. As we mention above, an assumption on $\beta$ is in general necessary because when $\beta$ is high enough logit dynamics roughly behaves as best-response dynamics. Moreover, in this case, the only metastable distributions have to be concentrated around the set of Nash equilibria. This is because for $\beta$ very high, it is extremely unlikely that a player leaves a Nash equilibrium. Then, the hardness results about the convergence of best-response dynamics for potential games, cf. e.g. [14], imply that the convergence to metastable distributions for high $\beta$ is similarly computationally hard. Interestingly, this difference in the behavior of the logit dynamics for different values of $\beta$ suggests that “the more noisy the system is, the more (meta)stable it is.”

Our result is in a sense existential, since it is unpractical to explicitly describe the distributions via the execution of Algorithm [4.1]. It is then an interesting open problem to characterize the sets $R_i$’s and $T_i$’s returned by this algorithm for some specific class of games in order to understand better the stability guarantee of the distributions. (We give a first example of this approach in Section 5.) A better understanding of spectra of the transition matrix along the lines of the results we prove may help in answering some of the questions above.

Naturally, there are other questions of general interest about metastability that we do not consider. For example, akin to price of anarchy and price of stability, one may ask what is the performance of a system in a metastable distribution? One might also want to investigate metastable behavior of different dynamics, such as best-response dynamics. However, in the latter case, no matter what selection rule is used to choose which player has to move next, a profile is never visited twice in time since at each step the potential goes down. Therefore, the “transient” behavior of best-response dynamics would roughly correspond to an (possibly exponentially long) sequence of profiles visited. This, however, would not add much to our understanding of the transient phase of best-response dynamics.

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References


A Hitting time tools

Consider a reversible Markov chain with state space $S$ and transition matrix $P$. For $L \subseteq S$ let $P_L, \lambda^L_t$ and $\lambda^L_{\text{max}}$ as defined in Section 4.5.1. Here we give a well known (see, e.g., [23]) variational characterization of $\lambda^L_{\text{max}}$ as expressed by the following lemma.

**Lemma A.1.** Consider a reversible Markov chain with state space $S$, transition matrix $P$ and stationary distribution $\pi$. For any $L \subseteq S$ we have

$$1 - \lambda^L_{\text{max}} = \inf_{\varphi \neq 0} \frac{\mathcal{E}_P(\varphi)}{\mathbb{E}_\pi[\varphi^2]},$$

where $\mathcal{E}_P(\varphi)$ is defined as in (7), $\mathbb{E}_\pi[\varphi^2] = \sum_x \pi(x)\varphi^2(x)$ and the inf is taken over functions $\varphi$ such that $\varphi(x) = 0$ for $x \in S \setminus L$ and $\mathbb{E}_\pi[\varphi^2] \neq 0$.

Since the statement of Lemma 3.10 is slightly different from the ones found in previous literature, we attach a proof for sake of completeness.

**Proof of Lemma A.1** Let $\varphi_L$ be the characteristic function on $L$, that is $\varphi_L(x) = 1$ if $x \in L$ and 0 otherwise. Then

$$P_x(\tau_{S\setminus L} > t) = \sum_{y \in S} P^t_L(x, y) = \sum_{y \in S} P^t_L(x, y)\varphi_L(y) = (P^t_L\varphi_L)(x). \quad (10)$$

Since $P_L$ is reversible with respect to $\pi_L$, we have that its eigenvectors, $\psi_1, \ldots, \psi|S|$, form an orthonormal basis with respect to the inner product $\langle \cdot, \cdot \rangle_{\pi_L}$: in particular we can write $\varphi_L = \sum_i \alpha_i \psi_i$, where $\sum \alpha_i = 1$ and each $\alpha_i > 0$. Hence and from the linearity of the inner product we have

$$\langle P^t_L\varphi_L, P^t_L\varphi_L \rangle_{\pi_L} = \sum_i \sum_j (\alpha_i \lambda_i^t \psi_i, \alpha_j \lambda_j^t \psi_j)_{\pi_L} \quad \text{(by orthogonality)}$$

$$\leq \left( \lambda^t_{\text{max}} \right)^{2t} \langle \varphi_L, \varphi_L \rangle_{\pi_L} = \left( \lambda^t_{\text{max}} \right)^{2t} \quad \text{(11)}$$

where the last equality follow from the definition of $\varphi_L$. Moreover,

$$\pi_L(x)[(P^t_L\varphi_L)(x)]^2 \leq \sum_{y \in S} \pi_L(y)(P^t_L\varphi_L)(y)^2 = \langle P^t_L\varphi_L, P^t_L\varphi_L \rangle_{\pi_L}. \quad (12)$$

The theorem follows from (10), (11), (12).

B Markov chain coupling

A coupling of two probability distributions $\mu$ and $\nu$ on a state space $S$ is a pair of random variables $(X, Y)$ defined on $S \times S$ such that the marginal distribution of $X$ is $\mu$ and the marginal distribution of $Y$ is $\nu$. A coupling of a Markov chain $M$ on $S$ with transition matrix $P$ is a process $(X_t, Y_t)_{t=0}^\infty$ with the property that $X_t$ and $Y_t$ are both Markov chains with transition matrix $P$. Similarly, a coupling of Markov chains $M, \bar{M}$ both defined on $S$ with transition matrices $P$ and $\bar{P}$, respectively, is a process $(X_t, Y_t)_{t=0}^\infty$ with the property that $X_t$ is a Markov chain with transition matrix $P$ and $Y_t$ is a Markov chain with transition matrix $\bar{P}$. When the two coupled chains start at $(X_0, Y_0) = (x, y)$, we write $P_{x,y}(\cdot)$ for the probability of an event on the space $S \times S$. The following theorem, which follows from Proposition 4.7 and Theorem 5.2 in [20] establishes the importance of this tool.
Theorem B.1 (Coupling). Let $\mathcal{M}, \tilde{\mathcal{M}}$ be two Markov chains with finite state space $S$ and transition matrices $P$ and $\tilde{P}$, respectively. For each pair of states $x, y \in S$ consider a coupling $(X_t, Y_t)$ of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ with starting states $X_0 = x$ and $Y_0 = y$. Then

$$\|P_t(x, \cdot) - \tilde{P}_t(y, \cdot)\|_{TV} \leq P_{x,y}(X_t \neq Y_t).$$