Designing Frugal Best-Response Mechanisms for Social Network Coordination Games

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Abstract

Social coordination games have recently received a lot of attention since they model several kinds of interaction problems in social networks. However, the performance of these games at equilibrium may be very bad. This motivates the adoption of mechanisms for inducing a socially optimal state. In this work we consider the design of incentive-compatible best-response mechanisms (Nisan, Schapira, Valiant, Zohar, 2011) for social coordination games. Specifically, we would like to compute special fees that may be assigned to players in order to induce the optimum profile of a social coordination game. Moreover, we would like the mechanism to be frugal, that is it can be implemented without any cost.

We show that a frugal incentive-compatible best-response mechanism for inducing the optimal profile of a two-strategy social coordination game always exists. Moreover, for such a mechanism, we investigate other properties inspired by envy-freeness, collusion-resistance and fairness. Finally, we show that extensions of the above results to other classes of games or to non-optimal profiles may be hard.

1 Introduction

In the last decade, *social networks* have been largely accepted as the prominent model to represent complex systems consisting of interconnected components. Game Theory has been widely adopted for the analysis of phenomena regarding social networks: examples can be found in Economics (see [GGJ$^+$10] and the several books devoted to this topic as, for example, [Goy07, Jac08, EK10]), in Biology (especially relevant is the work of Nowak [LHN05, Now06]), in Physics (see, for example, [GW10]) and in the Computer Science literature about online social networks (see, for example, [KKT05] and the rich set of works citing it). In many of these models, components interact only locally. That is, components, or *agents*, interact only with their neighbors on the social network and the relationship between them can be modeled as simple two-player games. We will refer to this model as a *social network game*.

Among social network games, particular interest has been given to *Social Coordination Games (SCGs)* (see Section 3 for a formal definition). In these games, the agents prefer to coordinate with their neighbors rather than conflicting with them. SCGs have been adopted, for example, in the study of the ferromagnetic Ising model [GW10], for modeling the diffusion of innovations [You98, You00, MS09] or the formation of opinions [BKO11, FGV12]. In Section 1.1 we give other remarkable applications that can be modeled by this class of games.

A specific feature of many social network games is given by the twofold nature of cost functions: in addition to costs arising from the relationship with neighbors on the social network, adopting a strategy may also have a cost that does not depend on what other players are doing. For example, in ferromagnetic Ising model [Mar99] this cost is given by the presence of magnetic fields; in opinion games [BKO11, FGV12] this cost models the personal belief of players; real-world applications described in Section 1.1 give other examples of these *preference costs*. Often these costs model the personal "feeling" of agents or their interaction with the environment. Hence they can also be seen as fixed costs attached to specific strategies. Moreover, costs arising from the social relationships and preference costs may also have different nature: usually, the former are monetary costs, whereas the latter are non-monetary. Nevertheless, it is common to represent the mixture of two cost components through a sum (as we do), even if they have different nature: a remarkable example of this approach is given by the famous network formation model given in [ADK$^+$08].

Unluckily, the analysis of SCGs shows that the self-interested behavior of agents may worsen the performance of the system. Specifically, in [FGV12] it has been shown that the *Price of Anarchy* (PoA) [KP09] for SCGs can be unbounded. Moreover, in these games the optimum is not necessarily a Nash equilibrium (and thus the Price of Stability [ADK$^+$08] is larger than 1) as shown in the following example.

Example 1. *Consider two players, both with strategy set* $\{0, 1\}$*, involved in the (social) coordination game specified by the following cost matrix:*

	0	1
0	$0, \varepsilon$	$1-\varepsilon, 1-\varepsilon$
1	$1-\varepsilon, 1-\varepsilon$	$\varepsilon, 0$

Assume moreover that the row player incurs a preference cost of $b \in \{0, 1\}$ *for playing strategy* b *while the column player's preference cost for playing strategy* b *is* $1 - b$*. The actual costs faced by the players are then as follows:*

	0	1
0	$0, 1+\varepsilon$	$1-\varepsilon, 1-\varepsilon$
1	$2-\varepsilon, 2-\varepsilon$	$1+\varepsilon, 0$

Thus, for $0 < \varepsilon < \frac{1}{3}$*, the two configurations of minimal total cost are* $(0, 0)$ *and* $(1, 1)$ *but the unique Nash equilibrium of the game is* $(0, 1)$*.*

However, in this work we will show (see Theorem 2) that if each player has exactly two strategies, then it is easy for a centralized algorithm to optimize the performance of these systems.

The occurrence of these two events, large PoA and efficient centralized optimization, suggests to design mechanisms able to influence the players' behavior towards the desired direction. Several and different mechanisms of this kind have been proposed, including *taxes* [BMW59, CDR03], *Stackelberg strategies* [KLO97], *mechanisms via creditability* [MT03] and *coordination mechanisms* [CKN09].

In this work we focus on *incentive-compatible best-response mechanisms.* This is a class of indirect mechanisms introduced in [NSVZ11] in which agents repeatedly play a base game and at each time step they are prescribed to choose the best-response to the strategies currently selected by other agents. Roughly speaking, this class of mechanisms takes advantage of the dynamical nature of many systems (see, for example, the real-world examples in Section 1.1) to induce the desired outcome.

In [NSVZ11] it is shown that for a specific class of games, namely *NBR-solvable games with clear outcomes* (roughly speaking, these are games for which a Nash equilibrium can be computed by iterated elimination of "useless" strategies; see Section 2.2 for a detailed definition), players have no incentive to deviate from this prescribed behavior and the mechanism converges to an equilibrium. Thus, for inducing the optimum of an SCG, it is sufficient to modify conveniently the players' cost functions so that the SCG becomes an NBR-solvable game with clear outcomes.

However, two constraints should be satisfied. On one side, the new cost functions should not worsen the performance of the desired profile. Indeed, it does not make sense to induce the profile that minimizes the social cost and then to have players paying more than this minimum. On the other side, the new cost functions should not avoid the players pay their preference costs. Indeed, as suggested above, these often are fixed costs of strategy depending only on personal or environmental features that cannot be influenced by neither other players nor any external authority.

In this work, we consider a special way of modifying a cost function. We assume that the mechanism may assign to players playing the desired strategy special *fees* (possibly negative) in place of the costs arising from social relationships. For example, if in the setting of Example 1 we would like to induce the optimal profile $(0, 0)$, then the mechanism may offer to i_1 a fee $-\delta$, with $\delta > \varepsilon$, whenever she plays strategy 0. That is, the players face this new game (the payoffs in the matrix do not include preference costs):

	0	1
0	$0, -\delta$	$1-\varepsilon, 1-\varepsilon$
1	$1-\varepsilon, -\delta$	$\varepsilon, 0$

It is easy to see that, by considering the preference costs described in Example 1, $(0, 0)$ is the unique Nash equilibrium, its cost is lower than in the original game and, moreover, the new game is solvable by iterated elimination of dominated strategies (a subclass of NBR-solvable games): playing 0 is a dominant strategy for i_0, and then (given that i_0 plays 0) playing 0 is a dominant strategy for i_1.

Implementing this mechanism has a cost $\delta + \varepsilon$ for the mechanism. Indeed, it should be necessary not only to pay δ to player i_1, but also to pay the communication costs of ε in her place. However, in this work we focus on *frugal* mechanisms, i.e. on mechanisms that can be implemented by a designer without any cost. This means that whenever inducing a player to play the target strategy has a cost for the designer, it should be possible to collect the necessary amount of money from other players (see Section 2.3 for a detailed definition). For example, in the setting of Example 1, the designer may start by offering a fee of $\delta + \varepsilon$, with $\varepsilon < \delta < 1$, to i_0; then, after i_0 payed this fee, he may offer a fee of $-\delta$ to i_1. The resulting game would be

	0	1
0	$\delta+\varepsilon, -\delta$	$\delta+\varepsilon, 1-\varepsilon$
1	$1-\varepsilon, -\delta$	$\varepsilon, 0$

As above the payoffs in the matrix do not include preference costs, and by considering the ones described in Example 1, $(0, 0)$ is the unique Nash equilibrium and the game is solvable by iterated elimination of dominated strategies. However, now the mechanism can be implemented with no cost for the designer.

1.1 Real-world examples

In this section we describe two different real-world settings that can be easily modeled in the setting described above.

1.1.1 French academics pools

Let us introduce the following example, drawn from a (simplified version of a) real case occurring in French academics. It is constituted by a myriad of institutions, including only in Paris region 14 universities, dozens of engineer and business schools, ... For several (strategic, scientific, maybe political) reasons, former President Nicolas Sarkozy proposed a few years ago to group some of these institutions into large pools – and to finance them. Rather quickly, several proposals came out, and let us focus on two important projects of pools in Paris region: PSL[1], located inside Paris, and UPSA[2], located in the suburbs.

This setting can be easily modeled through a SCG. Let us consider a set $N = \{1, 2, \dots, n\}$ of institutions: each institution may decide to join PSL or UPSA[3]. For an institute i, there is a personal cost to join one particular pool that does not depend on the institutions already in the pool (in particular, they arise from the necessity of changing location, since the pools shall be geographically coherent). Also, for each pair of institutions i and j, there is an interest of being in the same pool (more scientific cooperation, research project, common teaching program for instance).

The composition of the two pools was not fixed, and as a matter of fact some institutes decided over time to join one or the other one. Often, the the decision of institutions is made after a long bargaining. Thus, the formation of the two pool is inherently sequential. The sequentiality can be exploited by any authority interested in the creations of pools which are globally optimal, i.e., which minimizes the sum of the costs of institutions, as, for example, the Minister of Research and Education. In particular, the Minister, being the main financial contributor to these institutions (and to the pool), may convince a particular institution to join a particular pool by giving financial help (to change location) to the institution, or to increase the pool's endowment. And it will be clearly easier and less expensive to convince some institution to join a pool already containing many partners.

1.1.2 A network of firms

Another real-world application deals with the implementation of information sharing processes in networks of firms. An example is given by *RosettaNet* (`www.rosettanet.org`), a community of more than 500 international firms that share business information using standardized processes, with the objective

[1] Paris-Sciences-Lettres, `http://www.univ-psl.fr/`.
[2] University Paris-Saclay, `http://www.campus-paris-saclay.fr/`.
[3] Of course the real situation is much more complex, since they may stay on their own, or propose a new pool.

to improve the efficiency of transactions in increasingly complex supply chains [CO08]. RosettaNet is not the only community for the integration of business processes and business information exchanges, and a firm, accordingly to its commercial and technical features, may also decide to adopt another standard of communication (e.g., the Electronic data interchange (EDI)).

Again, the decisions taken by the firms can be modeled as an SCG. Indeed, for firms, adopting a standard, implies a faster coordination via shared processes and a more rapid exchange of information about business models. As a consequence, this stimulates new collaborations among firms and new opportunities of innovative business. On the other hand, for a firm, the activity of innovation demanded to enter in the community is costly, since it requires the adoption of new technologies (for instance, powerful hardware to run the recommended software) and new standards of common e-business language. In addition, to become a member of the community, a firm must pay a fee, whose cost depends on the kind of membership demanded (e.g. for the RosettaNet comunity, a firm pays 4500 U.S. Dollars if it wants to become a simple partner, till 80000 U.S Dollars if it wants to sit in the Global Council of the consortium [Wyc09]).

The formation of these communities is a dynamical process. In particular, the high cost related to their implementation in the business process of a firm has determined, at least in the first phase, a great adhesion of large companies. It is interesting to note that, recently, the membership rate of medium and small enterprises (MSEs) has risen. As above, an authority interested in inducing an optimal setting, for example the International Federation of Small & Medium Enterprises, can exploit this inherently sequential process. Indeed, it can support or sponsor different policies, such as the adoption of differentiated fees, the implementation of solutions dedicated to specific market segments or having large companies that serve as mentors to smaller firms with the goal to benefit from the opportunity to interface with their services [Wyc09].

1.2 Our contribution

The focus of this work is on designing frugal incentive-compatible best-response mechanisms for SCGs through the assignment of special fees to players in case they play the desired strategy.

We start by considering SCGs in which each player has two strategies. As stated above, the optimal profile can be efficiently computed for these games. Then, we show that it is always possible to efficiently design a frugal best-response mechanism for inducing this optimal profile (Theorem 3). Thus, an authority can always find policies that allow to exploit the dynamical nature of a system to induce the desired outcome.

Given this positive result, we investigate other desired properties that the mechanism may satisfy. The first property that we consider, named *order-freeness*, deals with the possibility that several frugal best-response mechanisms can be adopted for inducing the optimal profile. If these mechanisms treat one player in different ways, then this player will care about which mechanism is actually implemented. A frugal best-response mechanism is order-free if no player prefers that another mechanism is adopted. We will show that a frugal best-response mechanism that is order-free always exists (see Theorem 4). However, we also show that verifying if a mechanism satisfies this property is hard (see Theorem 5).

The second property on which we focus is the *collusion-resistance*. We would like that no coalition has any incentive to leave the induced profile even if side-payments are allowed. Interestingly, we show that a frugal best-response mechanism that is collusion-resistant always exists (see Theorem 6). Moreover, we prove a characterization of collusion-resistance property in terms of solutions of a suitable cooperative game.

Finally, we look for *fair* mechanisms. We give different definitions of fairness, that focus on different aspects of the problem. The first definition is based on the cooperative game characterization previously discussed. In this area, the Shapley value is uniformly recognized as a measure of fairness and, for this reason, we look at the extent to which we can adopt this concept in our setting. However, it will turn out that the Shapley value corresponds to a frugal best-response mechanism only for special SCGs (Theorem 7). For the other fairness definitions, we specify an ideal fair mechanism which we would like to be as close to as possible. Unfortunately, it will turn out that for each of the ideal mechanisms we considered, it is hard to compute the closest frugal best-response mechanism (see Theorem 8).

We conclude this work by trying to extend the previous results to other classes of games. However, we show that the optimal profile may be difficult to compute if we allow anti-coordination between players or if we allow more than two strategies for players. This negative finding motivates us to investigate the possibility of inducing non-optimal profiles (as, for example, profiles that are approximatively optimal): again, we face a negative result that rules out this possibility.

2 The Model

Throughout this paper we use bold symbols for vectors, i.e. $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Given a vector $\mathbf{x}$ of size n and a set $A \subseteq [n]$, we will write $\mathbf{x}_A$ for the vector $(x_i)_{i \in A}$ and $\mathbf{x}_{-A}$ for the vector $(x_i)_{i \notin A}$. We also use $\mathbf{x}_{-i}$ as a shorthand for $\mathbf{x}_{-\{i\}}$. Finally, for any $b \in \{0, 1\}$ we write $\bar{b}$ for $1 - b$.

2.1 Social Network Games

In an n-player *social network game* $\mathcal{G}$, each player $i \in [n]$ has strategy set S_i and is represented by a vertex of a *social graph* $G = (V, E)$. For each player i and each strategy $s_i \in S_i$, we denote by $p_i(s_i) \geq 0$ the cost for i in adopting s_i. To each edge $e = (i, j) \in E$ is linked a two player game $\mathcal{G}_e$ in which the set of strategies of the endpoints are exactly S_i and S_j. We denote by $c_i^e(s_i, s_j) \geq 0$ the cost for player i in the game $\mathcal{G}_e$, with $e = (i, j) \in E$, when i chooses the strategy $s_i \in S_i$ and j selects strategy $s_j \in S_j$. Given a strategy profile $\mathbf{x} \in S_1 \times \cdots \times S_n$, the total cost of player i in the social network game $\mathcal{G}$ is given by

$$c_i(\mathbf{x}) = p_i(x_i) + \sum_{e=(i,j)} c_i^e(x_i, x_j).$$

In this work we assume that costs $\{c_i^e\}_{i,e}$ are *communication costs*, whereas costs $\{p_i\}_i$ are *preference costs*. The former can be seen as costs that social relationship demands, whereas the latter are fixed costs independent of the social network.

2.2 Best-Response Mechanisms

Nisan et al. [NSVZ11] studied a class of indirect mechanisms, termed *repeated-response mechanisms*: starting from a given profile, at each time step t, some player i_t is selected and she announces a strategy $s_i^t \in S_i$. A *best-response mechanism* is a repeated-response mechanism in which the prescribed behavior for each player is to always choose a best-response to the strategies currently played by other players. A repeated-response mechanism *converges* to the target profile $\mathbf{x}$ if players eventually play according to this strategy profile, i.e. there is $t > 0$ such that $\mathbf{x}^{t'} = \mathbf{x}$ for any $t' \geq t$, where $\mathbf{x}^{t'}$ is the strategy profile after players selected at time step t announced their strategies. For a player i enrolled in a repeated-response mechanism, let us denote by $z_i(\mathbf{x}^t)$ the cost of i in the profile $\mathbf{x}^t$. If the mechanism converges to $\mathbf{x}$, we say that the *total cost* of i is $Z_i = z_i(\mathbf{x})$, otherwise we say the total cost of i is $Z_i = \limsup_{t \to \infty} z_i(\mathbf{x}^t)$. A repeated-response mechanism is *incentive-compatible* if any player behaving as prescribed by the mechanism achieves a total cost that is at most as high as the total cost achieved by deviating from the prescribed behavior, given that the other players play as prescribed. Specifically, a best-response mechanism is incentive-compatible if always choosing the best-response is a pure Nash equilibrium of the n-player game whose player's strategies are all possible behaviors and player's costs are their total costs.

We think that it is useful to remark that in [NSVZ11] (and in this work), players are only interested in minimizing the cost in the profile at which the best-response mechanism converges and they do not care about the cost in the intermediate profile generated by the dynamics. In some sense, the mechanism is run as a pre-computation for the actual play of the game. This is similar to running an ascending auction: the players do not pay their bid, but the auction format is used to compute the amount that should be paid and the best strategy for each player.

Nisan et al. [NSVZ11] showed that a best-response mechanism always converges and it is incentive-compatible for a specific class of games. Let us define this class. A strategy $s_i \in S_i$ is a *never-best-response (NBR) strategy* if for every $\mathbf{x}_{-i}$ there exists $s_i' \neq s_i$ such that $z_i(s_i, \mathbf{x}_{-i}) > z_i(s_i', \mathbf{x}_{-i})$. A game is *NBR-solvable* if there exists a sequence of eliminations of NBR strategies that results in a single strategy profile[4]. Let us denote by $\mathbf{y}^\star$ the unique profile to which the sequence of eliminations of an NBR-solvable game converges. For an NBR-solvable game $\mathcal{G}$ with a sequence of eliminations of length k, we denote

[4]In case of ties, we assume that an opportune tie-breaking rule is adopted.

by $\mathcal{G}_j$, with $j = 0, 1, \ldots, k$, the sub-game resulting from the first j eliminations in the sequence (observe that $\mathcal{G}_0 = \mathcal{G}$ and $\mathcal{G}_k = \mathbf{y}^\star$). Then an NBR-solvable game is said to have *clear outcomes* if for each player i, there exists a (player-specific) sequence of elimination of NBR strategies such that at the first step τ_i in which a strategy of S_i is eliminated, player i "likes" $\mathbf{y}^\star$ at least as much as any other profile in $\mathcal{G}_{\tau_i}$, i.e. $z_i(\mathbf{y}^\star) \leq z_i(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{G}_{\tau_i}$. The following theorem shows the relationship between this class of games and best-response mechanisms.

Theorem 1 ([NSVZ11])**.** *A best-response mechanism for an NBR-solvable game with clear outcomes is incentive-compatible and converges to $\mathbf{y}^\star$, regardless of the starting state and the order in which players are selected.*

Our aim is to design an incentive-compatible best-response mechanism for inducing a profile $\mathbf{y}$ in a social network game. That is, we want to compute a new cost function c'_i for each player i such that the resulting game $\mathcal{G}'$ is NBR-solvable with clear outcomes whose Nash equilibrium is $\mathbf{y}$. This is sufficient to implement the following mechanism:

Definition 1 (The mechanism)**.** *Let us rename players so that player i is the i-th player appearing in the elimination sequence defining $\mathcal{G}'$. Moreover, consider an arbitrary schedule of players (we allow that a player appears in the schedule more than one time or that more than one player is scheduled at the same time step). For each player i let t_i be the first time step i is scheduled, and before t_i, players $1, \ldots, i-1$ are scheduled at least once in this order. Then, at each time step t the mechanism assigns to a player i selected at time t a cost $c_i(\mathbf{x}^{t-1})$ if $t < t_i$ and a cost $c'_i(\mathbf{x}^{t-1})$ otherwise.*

Then, since $\mathcal{G}'$ is NBR-solvable with clear outcomes, by Theorem 1 at each time step players prefer to play according to the best-response and this dynamics converges to the target profile $\mathbf{y}$. Note that the time necessary for convergence depends on how players are scheduled for announcing their strategies. However, if we only consider schedules in which no player is "adversarially" delayed for arbitrarily long time, then the mechanism converges to the target profile quickly.

In our setting we force the new cost functions to satisfy two constraints. First, the social cost of the induced profile should not increase, i.e.

$$\sum_i c'_i(\mathbf{y}) \leq \sum_i c_i(\mathbf{y}), \tag{1}$$

where $\mathbf{y}$ is the target profile. Second, the new cost functions cannot avoid that players pay the preference cost linked to the target strategy. Indeed, these costs are fixed and independent of the network and cannot be influenced and modified by the mechanism.

In this work we focus on a special way of building these new cost functions. Formally, we consider a *vector of fees* $\boldsymbol{\gamma} = (\gamma(i))_{i\in[n]}$ and say that $\boldsymbol{\gamma}$, a social network game $\mathcal{G}$ and a strategy profile $\mathbf{y}$ *define* a game $\mathcal{G}_{\mathbf{y},\boldsymbol{\gamma}}$ in which the cost function c'_i of player i is as follows:

$$c'_i(\mathbf{x}) = \begin{cases} \gamma(i) + p_i(x_i), & \text{if } x_i = y_i; \\ c_i(\mathbf{x}), & \text{otherwise.} \end{cases}$$

Then, our aim becomes to compute a vector of fees $\boldsymbol{\gamma}$ such that the game $\mathcal{G}_{\mathbf{y},\boldsymbol{\gamma}}$ defined by $\mathcal{G}, \mathbf{y}$ and $\boldsymbol{\gamma}$ is NBR-solvable with clear outcomes.

2.3 Frugality

Consider a social network game $\mathcal{G}$, a profile $\mathbf{y}$, a vector of fees $\boldsymbol{\gamma}$ and the corresponding game $\mathcal{G}_{\mathbf{y},\boldsymbol{\gamma}}$. If we denote by c'_i the cost function of player i in $\mathcal{G}_{\mathbf{y},\boldsymbol{\gamma}}$, then the *cost* of $\boldsymbol{\gamma}$ is defined as

$$\sum_i (c_i(\mathbf{y}) - c'_i(\mathbf{y})) = C(\mathbf{y}) - \sum_i \gamma(i),$$

where

$$C(\mathbf{y}) = \sum_i \sum_{e=(i,j)} c_i^e(y_i, y_j).$$

Note that, by (1), the cost of a vector of fees should be always non-negative.

We are interested in designing incentive-compatible best-response mechanisms that are *frugal*, i.e. that have no cost at all for the mechanism. Obviously, this means that the vector of fees should have *null cost*.

However, this may be not sufficient. Indeed, some vector of fees allows that a mechanism gives an incentive to some players (in order that they play the target strategy) before that an equivalent amount of money has been collected from other players. We would like to avoid the occurrence even of this "temporary deficit". That is, we would like that the mechanism can schedule the players so that she is able to pay a player i with money collected from players scheduled before i. Specifically, let an *order of players* be a permutation π on the set of players. Then, we say that a vector of fees $\boldsymbol{\gamma}$ is *deficit-free according to* π if for each player i

$$\gamma(i) + \sum_{j \in N_\pi(i)} \gamma(j) \geq 0,$$

where $N_\pi(i)$ is the neighbors of i that are scheduled before i in π, i.e. $N_\pi(i) = \{j : e = (i,j) \in E \text{ and } \pi(j) < \pi(i)\}$. A vector of fees $\boldsymbol{\gamma}$ is *deficit-free* if there exists at least one order π of players such that $\boldsymbol{\gamma}$ is deficit-free according to π.

Note that deficit-freeness implies also that the designer has not a negative cost even in case of non-convergence. That is, the cost of the fees will always be non-negative for the designer no matter what the players will play and not only if the player play according the target profile.

3 Two-Strategy Social Coordination Games

Here, we consider the following subclass of social network games, named *two-strategy SCGs*, where each player has only two strategies, 0 and 1 and for every edge $e = (i,j)$ the game $\mathcal{G}_e$ is given by the following cost matrix:

	0	1
0	$\alpha_i^e(0), \alpha_j^e(0)$	$\beta_i^e(0), \beta_j^e(1)$
1	$\beta_i^e(1), \beta_j^e(0)$	$\alpha_i^e(1), \alpha_j^e(1)$

where the costs α_i^e for agreements are smaller or equal to the costs β_i^e for disagreements, i.e. $\beta_k^e(b) \geq \alpha_k^e(b') \geq 0$ for all b, b' and $k = i, j$. Roughly speaking, players like to agree with their neighbors rather than disagree, but the costs may vary depending on which strategy a player adopts.

We say that a strategy profile $\mathbf{x}^\star$ is *optimal* for a social network game if it minimizes $\sum_i c_i(\mathbf{x})$ over all profiles $\mathbf{x}$. The following theorem shows that the optimal profile can be easily computed in two-strategy SCGs.

Theorem 2. *The optimal profile of a two-strategy SCG is computable in polynomial time.*

Proof. For the sake of readability, we set $A_e = \alpha_i^e(0) + \alpha_j^e(0)$, $B_e = \beta_i^e(0) + \beta_j^e(1)$, $C_e = \beta_i^e(1) + \beta_j^e(0)$ and $D_e = \alpha_i^e(1) + \alpha_j^e(1)$.

We reduce the problem of computing the optimal profile to the problem of computing the minimum (s,t)-cut in a directed graph $G' = (V', E')$. The graph G' contains all the vertices of the social graph G underlying the game and 2 further nodes, named 0 and 1. Moreover, for each $e = (i,j)$ of the social graph, we add in G' four directed edges: $\overrightarrow{e} = \overrightarrow{(i,j)}$ and $\overleftarrow{e} = \overleftarrow{(i,j)}$ with weights $W_{\overrightarrow{e}} = B_e - A_e$ and $W_{\overleftarrow{e}} = C_e - D_e$, respectively; and $\overrightarrow{e_0} = \overrightarrow{(0,i)}$ and $\overrightarrow{e_1} = \overrightarrow{(i,1)}$ with weights $W_{\overrightarrow{e_0}} = D_e$ and $W_{\overrightarrow{e_1}} = A_e$, respectively. Finally, for each player i we add a directed edge $\overrightarrow{i_0} = \overrightarrow{(0,i)}$ of weight $W_{\overrightarrow{i_0}} = p_i(1)$ and a directed edge $\overrightarrow{i_1} = \overrightarrow{(i,1)}$ of weight $W_{\overrightarrow{i_1}} = p_i(0)$. Obviously, multiple edges between the same pair of nodes can be merged, by simply summing their weights. However, for the sake of understandability we assume throughout the proof that they are separate.

Consider the following one-to-one correspondence between a profile $\mathbf{x}$ of the game and a $(0,1)$-cut in the graph above described: $x_i = 0$ if i is on the 0-side of the cut and $x_i = 1$ otherwise. The cost of a profile $\mathbf{x}$ is $H(\mathbf{x}) = \sum_i c_i(\mathbf{x}) = \sum_{e=(i,j)} (c_i^e(x_i, x_j) + c_j^e(x_i, x_j)) + \sum_i p_i(x_i)$. We will show that the corresponding cut costs exactly $H(\mathbf{x})$. We start by showing that for each edge (i,j), its contribution to the cut is exactly the contribution of (i,j) to $H(\mathbf{x})$. We can distinguish four cases:

- if $x_i = x_j = 0$, then the contribution of edge $e = (i, j)$ to $H(\mathbf{x})$ is A_e; on the othe side, among all edges of G' corresponding to e, only the edge $\overrightarrow{e_1} = \overrightarrow{(i, 1)}$ belongs to the corresponding cut and its weight is exactly A_e;

- if $x_i = x_j = 1$, then the contribution of edge $e = (i, j)$ to $H(\mathbf{x})$ is D_e; on the othe side, among all edges of G' corresponding to e, only the edge $\overrightarrow{e_0} = \overrightarrow{(0, i)}$ belongs to the corresponding cut and its weight is exactly D_e;

- if $x_i = 0$ and $x_j = 1$, then the contribution of edge $e = (i, j)$ to $H(\mathbf{x})$ is B_e; on the othe side, among all edges of G' corresponding to e, the edges $\overrightarrow{e} = \overrightarrow{(i, j)}$ and $\overrightarrow{e_1} = \overrightarrow{(i, 1)}$ belong to the corresponding cut and their total weight is exactly B_e;

- if $x_i = 1$ and $x_j = 0$, then the contribution of edge $e = (i, j)$ to $H(\mathbf{x})$ is C_e; on the othe side, among all edges of G' corresponding to e, the edges $\overleftarrow{e} = \overleftarrow{(i, j)}$ and $\overrightarrow{e_0} = \overrightarrow{(0, i)}$ belong to the corresponding cut and their total weight is exactly C_e.

Finally, we show that for each player i, her contribution to the cut is exactly the contribution of i to $H(\mathbf{x})$. Indeed, if $x_i = 0$, then the contribution of player i is $p_i(0)$ and, among all edges of G' corresponding to i, only the edge $\overrightarrow{i_1} = \overrightarrow{(i, 1)}$ belongs to the cut and its weight is exactly $p_i(0)$. Similarly, if $x_i = 1$, then the contribution of player i is $p_i(1)$ and, among all edges of G' corresponding to i, only the edge $\overrightarrow{i_0} = \overrightarrow{(0, i)}$ belongs to the cut and its weight is exactly $p_i(1)$. □

3.1 A Best-Response Mechanism

Note that, given a two-strategy social network game $\mathcal{G}$ and a profile $\mathbf{x}$, the game $\mathcal{G}_{\mathbf{x},\boldsymbol{\gamma}}$ defined by $\mathcal{G}, \mathbf{x}$ and a vector of fees $\boldsymbol{\gamma}$ is NBR-solvable with clear outcomes if and only if, in $\mathcal{G}_{\mathbf{x},\boldsymbol{\gamma}}$, it is possible to schedule players so that, for each player i, choosing $x_i^\star$ is the unique best-response (under opportune tie-breaking) given that players scheduled before i are playing according to $\mathbf{x}$. Formally, given an order π, we say that a vector of fees $\boldsymbol{\gamma}$ is *inducing* a profile $\mathbf{x}$ *according to* π if for each i

$$\gamma(i) \leq \Gamma_\pi(i) := \sum_{\substack{e=(i,j)\\ j \in N_\pi(i)}} c_i^e(\overline{x}_i, x_j) + \sum_{\substack{e=(i,j)\\ j \notin N_\pi(i)}} \min_{b \in \{0,1\}} c_i^e(\overline{x}_i, b) + p_i(\overline{x}_i) - p_i(x_i).$$

Indeed if i plays $\overline{x}_i$, in addition to her preference cost $p_i(\overline{x}_i)$, for an edge $e = (i, j)$ she will pay for sure $c_i^e(\overline{x}_i, x_j)$ if $j \in N_\pi(i)$ and at least $\min_{b \in \{0,1\}} c_i^e(\overline{x}_i, b)$ if $j \notin N_\pi(i)$. We say that a vector of fees $\boldsymbol{\gamma}$ is *inducing* the profile $\mathbf{x}$ if there exists at least one order π such that $\boldsymbol{\gamma}$ is inducing $\mathbf{x}$ according to π. Then, the game $\mathcal{G}_{\mathbf{x},\boldsymbol{\gamma}}$ is NBR-solvable with clear outcomes if and only if $\boldsymbol{\gamma}$ is inducing $\mathbf{x}$.

Henceforth, we will focus on the second, more concise, property. We also write that an order π is *feasible* for a profile $\mathbf{x}$ if there exists a vector of fees of null cost that is both inducing $\mathbf{x}$ and deficit-free according to π. Similarly, a vector of fees $\boldsymbol{\gamma}$ is *valid* for $\mathbf{x}$ *according to* π if it makes π feasible for $\mathbf{x}$. Finally, a vector of fees $\boldsymbol{\gamma}$ is *valid* for $\mathbf{x}$ if there is an order π such that $\boldsymbol{\gamma}$ is valid for π according to $\mathbf{x}$.

Given these preliminary definitions, we show that a frugal incentive-compatible best-response mechanism for the optimal profile of a two-strategy SCG can be easily designed. Given such a game and its optimal profile $\mathbf{x}^\star$, we define the *base value* $B(i)$ of a player i as

$$B(i) = p_i(\overline{x}_i^\star) - p_i(x_i^\star) + \sum_{e=(i,j)} \alpha_i^e(\overline{x}_i^\star).$$

That is, $B(i)$ is the maximum fee that may be assigned to i if she is the first to be scheduled in an order according to which a vector of fees is inducing $\mathbf{x}^\star$. Observe that, since the maximum fee assignable to a player only increases if that player is scheduled later, we get that

$$\Gamma_\pi(i) \geq B(i) \tag{2}$$

for each player i and each order π. Moreover, we can prove the following lemma.

Lemma 1. *Given a two-strategy SCG and its optimal profile* $\mathbf{x}^\star$, $\sum_i B(i) \geq C(\mathbf{x}^\star)$.

Proof. We say that a connected subset U of players is *monochromatic* if all players of U play the same strategy in $\mathbf{x}^\star$. A monochromatic subset U is *maximal* if for each edge (i,j) with $i \in U$ and $j \notin U$ we have $x_i^\star \neq x_j^\star$. We denote by $\mathcal{U}$ the set of all maximal monochromatic subsets of players (note that $\mathcal{U}$ is a partition of the player set).

Now fix a maximal monochromatic subset U. Moreover, consider the optimum profile $\mathbf{x}^\star$ and the profile $(\overline{\mathbf{x}}_U^\star, \mathbf{x}_{-U}^\star)$. We observe that

$$\begin{aligned}
0 \geq \sum_i \left(c_i(\mathbf{x}^\star) - c_i(\overline{\mathbf{x}}_U^\star, \mathbf{x}_{-U}^\star)\right) &= \sum_{\substack{e=(i,j)\\ i,j\in U}} \left(\alpha_i^e(x_i^\star) + \alpha_j^e(x_j^\star)\right) + \sum_{\substack{e=(i,j)\\ i\in U, j\notin U}} \left(\beta_i^e(x_i^\star) + \beta_j^e(x_j^\star)\right)\\
&\quad - \sum_{\substack{e=(i,j)\\ i,j\in U}} \left(\alpha_i^e(\overline{x}_i^\star) + \alpha_j^e(\overline{x}_j^\star)\right) - \sum_{\substack{e=(i,j)\\ i\in U, j\notin U}} \left(\alpha_i^e(\overline{x}_i^\star) + \alpha_j^e(x_j^\star)\right)\\
&\quad + \sum_{i\in U} p_i(x_i^\star) - \sum_{i \in U} p_i(\overline{x}_i^\star)\\
&\geq \sum_{i\in U} \sum_{e=(i,j)} c_i^e(x_i^\star, x_j^\star) - \sum_{i\in U} B(i),
\end{aligned}$$

where in the last inequality we use $\beta_j^e(x_j^\star) - \alpha_j^e(x_j^\star) \geq 0$ for $j \notin U$. Hence, $\sum_i B(i) = \sum_{U\in\mathcal{U}} \sum_{i\in U} B(i) \geq \sum_{U\in\mathcal{U}} \sum_{i\in U} \sum_{e=(i,j)} c_i^e(x_i^\star, x_j^\star) = C(\mathbf{x}^\star)$. □

Now we are ready to prove the existence of a frugal incentive-compatible best-response mechanism.

Theorem 3. *For a two-strategy SCG, a vector of fees valid for $\mathbf{x}^\star$ always exists and can be computed in polynomial time.*

Proof. Consider the following vector of fees $\boldsymbol{\gamma}$: for each i such that $B(i) \geq 0$, we set $\gamma(i) \leq B(i)$ such that

$$\sum_{i\colon B(i)\geq 0} \gamma(i) = C(\mathbf{x}^\star) - \sum_{i\colon B(i)<0} B(i).$$

By Lemma 1 this is always possible. Moreover, for remaining players i, we set $\gamma(i) = B(i)$. Note that the vector $\boldsymbol{\gamma}$ has null cost by construction.

By (2), this set of fees is inducing according to any order π. In particular, consider the order π that schedule first all players i such that $\gamma(i) \geq 0$ and then the remaining players: it is immediate to see that $\boldsymbol{\gamma}$ is deficit-free according to π. □

Given this result, it is natural to ask if other properties may be satisfied by a vector of fees. Next, we focus on three natural and interesting properties: order-freeness, collusion-resistance and fairness.

3.2 Order-Freeness

Suppose that two different orders, π and π', are both feasible for $\mathbf{x}^\star$ and let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ be corresponding valid vectors of fees. Then, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ can charge very different costs to the same player, as shown in the following example.

Example 2. *Consider a two-strategy SCG with four players, i_0, i_1, i_2, i_3, on the social graph G containing edges $e_k = (i_k, i_{k+1})$ for $k = 0, \ldots, 3$ (indices are modulo 4) and the edge $e_\star = (i_0, i_2)$. For each $k = 0, \ldots, 3$, we set $\alpha_{i_k}^{e_k}(b) = \alpha_{i_{k+1}}^{e_k}(b) = \alpha > 0$ and $\beta_{i_k}^{e_k}(b) = \beta_{i_{k+1}}^{e_k}(b) = \alpha + 1$ for each $b \in \{0,1\}$. For the edge $e_\star$, we set $\alpha_{i_0}^{e_\star}(b) = \alpha_{i_2}^{e_\star}(b) = \alpha$ and $\beta_{i_0}^{e_\star}(b) = \beta_{i_2}^{e_\star}(b) = M \geq 10\alpha$ for each $b \in \{0,1\}$. Moreover, $p_{i_0}(0) = p_{i_1}(1) = p_{i_2}(0) = p_{i_3}(1) = 0$ and $p_{i_0}(1) = p_{i_1}(0) = p_{i_2}(1) = p_{i_3}(0) = 1$. Finally, consider the optimal profile $\mathbf{x}^\star$ in which each player chooses the alternative 0. Then the vector of fees $\boldsymbol{\gamma}$ that assigns $\gamma(i_k) = 0$ for $k = 0,1,3$ and $\gamma(i_2) = 10\alpha$ is valid for $\mathbf{x}^\star$ if players are scheduled in the order i_0, i_1, i_3, i_2. Similarly, the vector of fees $\boldsymbol{\gamma}'$ that assigns $\gamma'(i_k) = 0$ for $k = 1,2,3$ and $\gamma'(i_0) = 10\alpha$ is valid for $\mathbf{x}^\star$ if players are scheduled in the order i_2, i_1, i_3, i_0. Thus, player i_0 pays very different fees depending on which schedule is adopted.*

It may be questionable that a player accepts the proposed fee if there exists another feasible order for which she improves her utility for sure. Thus, we would like to have a vector of fees $\boldsymbol{\gamma}$ for which no player "envies" another schedule or feels disadvantaged by the mechanism. Formally, let Π denote the set of all orders feasible for $\mathbf{x}^\star$. Since player i can be charged up to $\Gamma_\pi(i)$ in order π (for having $\boldsymbol{\gamma}$ inducing $\mathbf{x}^\star$ according to π), choosing a fee $\gamma(i)$ which is at most $\Gamma_\pi(i)$ for *any* $\pi \in \Pi$ would ensure "envy-freeness" for player i. Then by defining $\boldsymbol{\Gamma}^\star = (\Gamma^\star(i))_i$ the vector that sets $\Gamma^\star(i) = \min_{\pi\in\Pi} \Gamma_\pi(i)$, we say that a vector of fees $\boldsymbol{\gamma}$ is *order-free valid* for $\mathbf{x}^\star$ if it is valid for $\mathbf{x}^\star$ and $\gamma(i) \leq \Gamma^\star(i)$ for each player i.

In Example 2, let $\alpha = 1/3$ to fix the ideas. With the vector of fees γ, i_2 may complain, arguing that if she was scheduled first, she would have been charged at most $1+3\alpha = 2$, instead of $10/3$ in the current situation. For i_0 and i_2 the best possibility is to be scheduled first, and $\Gamma^\star(i_0) = \Gamma^\star(i_2) = 1 + 3\alpha = 2$. For i_1 and i_3, the best possibility is to be scheduled second (they cannot be scheduled first otherwise deficit-freeness would be broken) and $\Gamma^\star(i_1) = \Gamma^\star(i_3) = 2\alpha = 2/3$.

Interestingly, order-free validity can always be achieved, as stated in the following theorem.

Theorem 4. *For a two-strategy SCG $\mathcal{G}$, a vector of fees that is order-free valid for $\mathbf{x}^\star$ always exists and it can be computed in polynomial time.*

Proof. By (2), $\Gamma_\pi(i) \geq B(i)$ for any order π, hence $\Gamma^\star(i) \geq B(i)$. So, the theorem easily follows by observing that the vector of fees $\boldsymbol{\gamma}$ described in the proof of Theorem 3 is order-free valid, since $\gamma(i) \leq B(i) \leq \Gamma^\star(i)$ for each player i. □

Beyond the existence of one order-free valid vector of fees, we might be interested in determining whether a given vector of fees is order-free valid or not (for instance if we are focused on *fair* vector of fees, see Section 3.5). Unfortunately, computing the values $(\Gamma^\star(i))_i$ is NP-hard, as shown in the following theorem.

Theorem 5. *Given a two-strategy SCG and a player i, establishing whether $\Gamma^\star(i) = B(i)$ is (strongly) NP-complete, even if $\alpha_i^e(0) = \alpha_i^e(1) = \alpha_i^e$ and $\beta_i^e(0) = \beta_i^e(1) = \beta_i^e$ for each player i and each edge e.*

Proof. For verifying the membership in NP we use an order π as a witness. Then, it is sufficient to verify that π is feasible for $\mathbf{x}^\star$ and that $\Gamma_\pi(i) = B(i)$.

As for the completeness, we will show a polynomial reduction from the problem of deciding if a *feedback arc set* of size at most M, exists in a direct graph $D = (U, H)$ [GJ79]. A feedback arc set is a subset of edges whose deletion removes all cycles in D. Given an instance of this problem, we build an SCG as follows: there are $|U| + 2$ players, one for each vertex of the graph D (we will say these players belong to U) and two additional players that we call i_0 and i_1. For each player $u \in U$ we set $p_u(0) = p_u(1) = 0$; for player i_0 we set $p_{i_0}(0) = |H| - M$ and $p_{i_0}(1) = 0$; for player i_1 we set $p_{i_1}(0) = 0$ and $p_{i_1}(1) = |H| - M + \varepsilon$, with $\varepsilon > 0$. As for the edges of the social graph, let us start by considering the graph $D^\star = (U, H^\star)$ obtained by *bi-directing* D. That is, for each edge $e = (u, v) \in H$, there is a corresponding edge $e = (u, v) \in H^\star$ with weight $W_e = 1$; moreover, if edge $(u, v) \in H$ and $(v, u) \notin H$, then we add the edge $e = (v, u) \in H^\star$ with weight $W_e = 0$. We are now ready to describe the graph $G = (U \cup \{i_0, i_1\}, E)$ underlying the SCG:

- for each pair of players $u, v \in U$, if there exists an edge joining them in $H^\star$, then we add the edge $e = (u, v)$ to E. Moreover, we set $\alpha_u^e = \alpha_v^e = 0$, $\beta_u^e = W_{(v,u)}$ and $\beta_v^e = W_{(u,v)}$;
- finally, we add the edge $e = (i_0, i_1)$ to E and we set $\alpha_{i_0}^e = \alpha_{i_1}^e = \beta_{i_1}^e = 0$ and $\beta_{i_0}^e > |H| - M + \varepsilon$.

Consider now the optimal profile $\mathbf{x}^\star = \mathbf{0}$ and focus on player i_0. We will show that if a feedback arc set of size at most M exists, then $\Gamma^\star(i_0) = B(i_0)$. Consider indeed the DAG $D' = (U, H')$ resulting from the elimination of the feedback arc set and let us build an order π as follows: we first schedule players from U in a topological ordering of vertices in D', then we schedule the player i_0 and, at the end, the player i_1. Observe that, $\Gamma_\pi(i_0) = B(i_0) = -(|H| - M)$ and $\Gamma_\pi(i_1) = 0$. Moreover, for each player $u \in U$, $\Gamma_\pi(u)$ is exactly the number of ingoing edges in D'. Thus $\sum_{u\in U} \Gamma_\pi(u) = |H'| \geq |H| - M$. Then, there exists a vector of fees $\boldsymbol{\gamma}$ such that $\gamma(i_0) = \Gamma_\pi(i_0)$, $\gamma(i_1) = \Gamma_\pi(i_1)$ and, for players $u \in U$, we have that $0 \leq \gamma(u) \leq \Gamma_\pi(u)$ and $\sum_{u\in U} \gamma(u) = |H| - M$. Obviously, $\boldsymbol{\gamma}$ is inducing according to π. Thus, since i_0 is the unique player with a negative fee, $\boldsymbol{\gamma}$ is deficit-free according to π. Finally, $\boldsymbol{\gamma}$ has null cost since $C(\mathbf{x}^\star) = 0 = \sum_k \gamma(k)$. Thus, π is an order feasible for $\mathbf{x}^\star$ such that $\Gamma_\pi(i_0) = B(i_0)$ and hence $\Gamma^\star(i_0) = B(i_0)$.

We conclude the proof, by showing that if $\Gamma^\star(i_0) = B(i_0)$, then a feedback arc set of size at most M exists. Since $\Gamma^\star(i_0) = B(i_0)$, there exists an order π feasible for $\mathbf{x}^\star$ such that $\Gamma_\pi(i_0) = B(i_0) = -(|H| - M)$. Given this order π, we observe that i_0 should be scheduled before i_1, otherwise $\Gamma_\pi(i_0) = B(i_0) + \beta_{i_0}^{(i_0,i_1)} > B(i_0)$. Moreover, we can build a DAG $D' = (U, H')$ from π and the graph D by considering in H' only edges $(u, v) \in H$ such that u is scheduled before v. Observe that, for each $u \in U$, $\Gamma_\pi(u) = \sum_{v \in N_\pi(u)} W_{(v,u)}$ is exactly the number of ingoing edges in D'. Thus, $\sum_{u \in U} \Gamma_\pi(u) = |H'|$. Let us consider now a vector of fees $\boldsymbol{\gamma}$ valid for $\mathbf{x}^\star$ according to π. Since $\boldsymbol{\gamma}$ is inducing for $\mathbf{x}^\star$ according to π, then $\gamma(i_0) \leq -(|H| - M)$ and $\sum_{u \in U} \gamma(u) \leq |H'|$. Moreover, since $\boldsymbol{\gamma}$ is deficit-free according to π and i_1 is scheduled after i_0, then $\gamma(i_0) + \sum_{u \in U} \gamma(u) \geq 0$. Hence, we have $|H| - |H'| \leq M$. That is, a DAG has been obtained by D by deleting at most M edges and thus a feedback arc set of size at most M exists. $\square$

3.3 Collusion-Resistance

In designing a best-response mechanism for inducing $\mathbf{x}^\star$, we implement a new game that makes this profile the unique Nash equilibrium. In this way, no player has an incentive to deviate from $\mathbf{x}^\star$. However, this does not prevent some players to collude and jointly move away from $\mathbf{x}^\star$. In this section, we wonder about the possibility of inducing a profile $\mathbf{x}^\star$ such that no coalition of players deviates from $\mathbf{x}^\star$, even if side payments are allowed. Specifically, we say that a vector of fees $\boldsymbol{\gamma}$ is *collusion-resistant* if for every subset $L \subset [n]$ of players and any joint strategy $\mathbf{y}_L \neq \mathbf{x}^\star_L$

$$\sum_{i \in L} \left(p_i(x_i^\star) + \gamma(i)\right) \leq \sum_{i \in L} \left(p_i(y_i) + h_i(y_i)\right),$$

where

$$h_i(y_i) = \begin{cases} \gamma(i), & \text{if } y_i = x_i^\star; \\ \sum_{\substack{e=(i,j) \\ j \in L}} c_i^e(y_i, y_j) + \sum_{\substack{e=(i,j) \\ j \notin L}} c_i^e(y_i, x_j^\star), & \text{otherwise.} \end{cases}$$

Roughly speaking, we would like to choose a vector of fees $\boldsymbol{\gamma}$ such that the cumulative cost of a coalition L is minimized by playing according to $\mathbf{x}^\star$ (and thus each of the members gets the corresponding fee from $\boldsymbol{\gamma}$), given that the other players are currently playing according to this optimal profile. Note that this definition is stronger than asking for a $\boldsymbol{\gamma}$ that implements a strong Nash equilibrium, since we allow utility transfer.

Despite the strength of the definition, we have the following theorem.

Theorem 6. *For a two-strategy SCG a collusion-resistant vector of fees order-free valid for $\mathbf{x}^\star$ always exists and can be computed in polynomial time.*

Proof. This theorem easily follows by observing that the vector of fees $\boldsymbol{\gamma}$ described in the proof of Theorem 3 is collusion-resistant. Indeed, for each coalition L and each joint strategy $\mathbf{y}_L \neq \mathbf{x}^\star_L$ we have that:

- for each player $i \in L$ such that $y_i = x_i^\star$, $p_i(x_i^\star) + \gamma(i) = p_i(y_i) + h_i(y_i)$;
- for each player $i \in L$ such that $y_i \neq x_i^\star$,

$$p_i(x_i^\star) + \gamma(i) \leq p_i(x_i^\star) + B(i) = p_i(y_i) + \sum_{e=(i,j)} \alpha_i^e(y_i) \leq p_i(y_i) + h_i(y_i). \quad \square$$

It is interesting to note that for any two-strategy game (not necessarily SCGs) we can give an alternative definition of collusion-resistance based on cooperative cost games (see Appendix A for a review of the main concepts about cooperative cost games).

We consider the following cost game v:

$$v(L) = \begin{cases} \sum_{i \in L} \left(p_i(\overline{x}_i^\star) - p_i(x_i^\star) + \sum_{\substack{e=(i,j) \\ j \in L}} c_i^e(\overline{x}_i^\star, \overline{x}_j^\star) + \sum_{\substack{e=(i,j) \\ j \notin L}} c_i^e(\overline{x}_i^\star, x_j^\star) \right), & \text{if } L \subset [n]; \\ C(\mathbf{x}^\star), & \text{otherwise.} \end{cases} \tag{3}$$

Then, we have the following characterization.

Lemma 2. *Given a two-strategy game and its optimal profile $\mathbf{x}^\star$, a vector of fees $\boldsymbol{\gamma}$ is collusion-resistant and has null cost if and only if $\boldsymbol{\gamma}$ is in the core of the cooperative cost game v defined by* (3).

Proof. If $\boldsymbol{\gamma}$ has null cost, then $\sum_i \gamma(i) = C(\mathbf{x}^\star) = v([n])$. Thus $\boldsymbol{\gamma}$ is an efficient solution. Moreover, for any coalition L, let us consider the joint strategy $\overline{\mathbf{x}}^\star_L$. If $\boldsymbol{\gamma}$ is collusion-resistant, then

$$\sum_{i\in L}\gamma(i) \le \sum_{i\in L}\left(p_i(\overline{x}^\star_i) - p_i(x^\star_i) + \sum_{\substack{e=(i,j)\\ j\in L}} c^e_i(\overline{x}^\star_i, \overline{x}^\star_j) + \sum_{\substack{e=(i,j)\\ j\notin L}} c^e_i(\overline{x}^\star_i, x^\star_j)\right) = v(L).$$

Thus $\boldsymbol{\gamma}$ is in the core of v.

Now, consider a solution $\boldsymbol{\psi}_v$ in the core of v. Then $\sum_i \psi_v(i) = v([n]) = C(\mathbf{x}^\star)$. Thus the vector of fees $\boldsymbol{\gamma} = \boldsymbol{\psi}_v$ has null cost. Given a coalition L and a joint strategy $\mathbf{y}_L$, we will denote by L' the set containing any member $i \in L$ such that $y_i \neq x^\star_i$. Then, since $\sum_{i\in L'} \psi_v(i) \le v(L')$, we have

$$\begin{aligned}
\sum_{i\in L}\left(p_i(x^\star_i) + \psi_v(i)\right) &= \sum_{i\in L\setminus L'}\left(p_i(x^\star_i) + \psi_v(i)\right) + \sum_{i\in L'}\left(p_i(x^\star_i) + \psi_v(i)\right)\\
&\le \sum_{i\in L\setminus L'}\left(p_i(x^\star_i) + \psi_v(i)\right)\\
&\quad + \sum_{i\in L'}\left(p_i(\overline{x}^\star_i) + \sum_{\substack{e=(i,j)\\ j\in L'}} c^e_i(\overline{x}^\star_i, \overline{x}^\star_j) + \sum_{\substack{e=(i,j)\\ j\notin L'}} c^e_i(\overline{x}^\star_i, x^\star_j)\right)\\
&= \sum_{i\in L}\left(p_i(y_i) + h_i(y_i)\right).
\end{aligned}$$

Thus $\boldsymbol{\gamma}$ is collusion-resistant. ☐

3.4 Shapley-fairness

In cooperative game theory the *Shapley value* [Sha53] is uniformly acknowledged as a measure of fairness in settings where transfer of utility is allowed. Thus, the characterization introduced in the previous section motivates us to adopt such a fairness measure also in our setting.

The Shapley value $\boldsymbol{\Phi}_v$ applied to a cost game v is given by

$$\Phi_v(i) = \sum_{L\subseteq[n]\setminus i} \frac{\ell!(n-\ell-1)!}{n!}\left(v(L\cup i) - v(L)\right) \tag{4}$$

for each $i \in [n]$, where ℓ is the cardinality of coalition L (the reader can refer to Appendix B for a list of useful properties of the Shapley value).

Then, given a two-strategy SCG and its optimal profile $\mathbf{x}^\star$, we say that a vector of fees $\boldsymbol{\gamma}$ is *Shapley-fair* if $\boldsymbol{\gamma}$ coincides with the Shapley value for the cooperative game defined by (3). The following lemma shows that a Shapley-fair vector of fees can be easily computed.

Lemma 3. *The Shapley value of the cooperative game defined by* (3) *is computable in polynomial time.*

Proof. Let us define the following characteristic functions:

$$v_1(L) = \sum_{i\in L}\left(p_i(\overline{x}^\star_i) - p_i(x^\star_i)\right);\ v_2(L) = \sum_{i\in L}\sum_{\substack{e=(i,j)\\ j\in L}} c^e_i(\overline{x}^\star_i, \overline{x}^\star_j);\ v_3(L) = \sum_{i\in L}\sum_{\substack{e=(i,j)\\ j\notin L}} c^e_i(\overline{x}^\star_i, x^\star_j);$$

$$v_4(L) = \begin{cases} C(\mathbf{x}^\star) - \sum_i \left(p_i(\overline{x}^\star_i) - p_i(x^\star_i) + \sum_{e=(i,j)} c^e_i(\overline{x}^\star_i, \overline{x}^\star_j)\right), & \text{if } L = [n];\\ 0, & \text{otherwise.}\end{cases}$$

It is immediate to see that $v = v_1 + v_2 + v_3 + v_4$, where v is defined as in (3). Thus the Shapley value $\boldsymbol{\Phi}_v$ of v is, by additivity, $\boldsymbol{\Phi}_v = \boldsymbol{\Phi}_{v_1} + \boldsymbol{\Phi}_{v_2} + \boldsymbol{\Phi}_{v_3} + \boldsymbol{\Phi}_{v_4}$.

Let us evaluate separately each one of these Shapley values. Note that v_1 is an additive game, i.e. $v_1(L) = \sum_{i \in L} v_1(i)$ for each coalition L. Then, the marginal contribution $v(L \cup i) - v(L)$, for each player i and all coalitions L not containing i, is exactly $p_i(\overline{x}_i^\star) - p_i(x_i^\star)$, and, by (4), it immediately follows that $\Phi_{v_1}(i) = v_1(\{i\}) = p_i(\overline{x}_i^\star) - p_i(x_i^\star)$ for each player i.

Now consider the game v_2. Note that this game can be written as a sum of unanimity games (see Appendix B). That is $v_2(L) = \sum_{\substack{e=(i,j)\\ i,j \in L}} u^{k_e}_{\{i,j\}}(L)$, for each nonempty coalition L, where $k_e = c_i^e(\overline{x}_i^\star, \overline{x}_j^\star) + c_j^e(\overline{x}_j^\star, \overline{x}_i^\star)$. The Shapley value of game $u^{k_e}_{\{i,j\}}(L)$ is $\frac{k_e}{2}$ for i and j, and 0 for all other players. Then, for each player $i \in [n]$, we have

$$\Phi_{v_2}(i) = \frac{1}{2} \sum_{e=(i,j)} (c_i^e(\overline{x}_i^\star, \overline{x}_j^\star) + c_j^e(\overline{x}_j^\star, \overline{x}_i^\star)).$$

For the third game v_3, first note that for each player i and each nonempty coalition $L \subseteq [n] \setminus i$

$$v_3(L \cup i) - v_3(L) = \sum_{\substack{e=(i,j)\\ j \notin L}} c_i^e(\overline{x}_i^\star, x_j^\star) - \sum_{\substack{e=(i,j)\\ j \in L}} c_j^e(\overline{x}_j^\star, x_i^\star),$$

and

$$v_3([n] \setminus L) - v_3([n] \setminus (L \cup i)) = \sum_{\substack{e=(i,j)\\ j \in L}} c_i^e(\overline{x}_i^\star, x_j^\star) - \sum_{\substack{e=(i,j)\\ j \notin L}} c_j^e(\overline{x}_j^\star, x_i^\star).$$

Moreover, for the empty coalition we have that $v_3(i) - v_3(\emptyset) = \sum_{e=(i,j)} c_i^e(\overline{x}_i^\star, x_j^\star)$, and $v_3([n]) - v_3([n]\setminus i) = -\sum_{e=(i,j)} c_j^e(\overline{x}_j^\star, x_i^\star)$. Hence and since in (4) the coefficient for L and $[n] \setminus (L \cup i)$ are the same, we have

$$\begin{aligned}
\Phi_{v_3}(i) &= \sum_{\substack{L \subseteq [n] \setminus \{i\}\\ \ell < n/2}} \frac{\ell!(n-\ell-1)!}{n!} (v_3(L \cup i) - v_3(L) + v([n] \setminus L) - v_3([n] \setminus (L \cup i)) \\
&= \sum_{0 \le \ell < \frac{n}{2}} \frac{\ell!(n-\ell-1)!}{n!} \binom{n-1}{\ell} \sum_{e=(i,j)} (c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star)) \\
&= \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \sum_{e=(i,j)} (c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star)),
\end{aligned}$$

for each player i.

Finally, observe that v_4 is non-zero only for the grand coalition. Then, by the property of symmetry of the Shapley value, each player receives the same fraction of the value of the grand coalition, i.e. for each player i

$$\Phi_{v_4}(i) = \frac{1}{n} \left(C(\mathbf{x}^\star) - \sum_k \left(p_k(\overline{x}_k^\star) - p_k(x_k^\star) + \sum_{e=(k,j)} c_k^e(\overline{x}_k^\star, \overline{x}_j^\star) \right) \right).$$

Thus, for the cost game v we have that for each player i

$$\begin{aligned}
\Phi_v(i) = p_i(\overline{x}_i^\star) - p_i(x_i^\star) &+ \frac{1}{2} \sum_{e=(i,j)} \left(c_i^e(\overline{x}_i^\star, \overline{x}_j^\star) + c_j^e(\overline{x}_j^\star, \overline{x}_i^\star) \right) \\
&+ \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \sum_{e=(i,j)} \left(c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star) \right) \\
&+ \frac{1}{n} \left(C(\mathbf{x}^\star) - \sum_k \left(p_k(\overline{x}_k^\star) - p_k(x_k^\star) + \sum_{e=(k,j)} c_k^e(\overline{x}_k^\star, \overline{x}_j^\star) \right) \right),
\end{aligned}$$

and the lemma follows. $\square$

Interestingly, the following lemma shows that the Shapley value always corresponds to a collision-resistant vector of fees of null cost.

Lemma 4. *For a two-strategy SCG, the Shapley value of the cooperative game defined by* (3) *is always in the core.*

Proof. The lemma follows by showing that for each one of the characteristic functions v_1, v_2, v_3 and v_4 defined in the proof of Lemma 3, the Shapley value is in the core.

Obviously, $\mathbf{\Phi}_{v_1} \in \texttt{core}(v_1)$. Since the Shapley value of unanimity games lies in the core (see Appendix B), then $\mathbf{\Phi}_{v_2} \in \texttt{core}(v_2)$. As for v_3, consider a nonempty coalition L. Then,

$$\begin{aligned}\sum_{i\in L}\Phi_{v_3}(i) &= \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\sum_{i\in L}\sum_{e=(i,j)}(c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star))\\ &\le \sum_{i\in L}\sum_{\substack{e=(i,j)\\ j\notin L}}(c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star))\\ &\le \sum_{i\in L}\sum_{\substack{e=(i,j)\\ j\notin L}} c_i^e(\overline{x}_i^\star, x_j^\star) = v_3(L).\end{aligned}$$

Thus, also $\mathbf{\Phi}_{v_3} \in \texttt{core}(v_3)$.

For v_4, consider a player k. Observe that

$$p_k(\overline{x}_k^\star) - p_k(x_k^\star) + \sum_{e=(k,j)} c_k^e(\overline{x}_k^\star, \overline{x}_j^\star) \ge p_k(\overline{x}_k^\star) - p_k(x_k^\star) + \sum_{e=(g,j)} \alpha_k^e(\overline{x}_k^\star) = B(k).$$

Thus, we have $\Phi_{v_4}(i) \le \frac{1}{n}\left(C(\mathbf{x}^\star) - \sum_k B(k)\right)$ for each player i. From Lemma 1, it follows that

$$\Phi_{v_4}(i) \le 0 \tag{5}$$

for each i and, consequently, $\sum_{i\in L}\Phi_{v_4}(i) \le v_4(L)$ for each coalition $L \subset [n]$. Then, since $\sum_i \Phi_{v_4}(i) = v_4([n])$, also $\mathbf{\Phi}_{v_4} \in \texttt{core}(v_4)$. □

Unfortunately, it is easy to find an example showing that the Shapley value may not be a valid vector of fees for the optimum of a social graph, even in the very simple setting in which for each edge $e = (i, j)$ we have $\alpha_i^e(0) = \alpha_i^e(1) = \alpha_j^e(0) = \alpha_j^e(1) = \alpha^e$ and $\beta_i^e(0) = \beta_i^e(1) = \beta_j^e(0) = \beta_j^e(1) = \beta^e$.

Example 3. *Consider five players, named* $1, 2, 3, 4, 5$*, with an underlying social graph consisting of four edges* $e_i = (i, i+1)$ *for* $i = 1, \ldots, 4$*. Let us fix a constant* $\beta > 0$*. We set* $p_1(0) = p_2(0) = p_3(1) = 0$*,* $p_1(1) = \frac{4}{5}\beta + \frac{\varepsilon}{5}$*,* $p_2(1) = \frac{6}{5}\beta + \frac{4}{5}\varepsilon$*,* $p_3(0) = 2\beta + \varepsilon$*, with* $0 < \varepsilon < \frac{\beta}{2}$ *and, finally,* $p_4(b) = p_5(b) = 0$ *for each* $b \in \{0, 1\}$*. Moreover, we set* $\alpha^e = 0$ *and* $\beta^e = \beta$ *for each edge* e*. It is easy to see that in the optimal profile* $\mathbf{x}^\star$ *players* 1 *and* 2 *choose strategy* 0 *and the remaining players select strategy* 1*. You can check the Shapley value* $\mathbf{\Phi}_v$ *sets* $\Phi_v(1) = -\frac{\varepsilon}{5}$*,* $\Phi_v(2) = \frac{7}{5}\beta + \frac{2}{5}\varepsilon$*,* $\Phi_v(3) = \frac{11}{5}\beta + \frac{3}{5}\varepsilon$*,* $\Phi_v(4) = \Phi_v(5) = -\frac{4}{5}\beta - \frac{2}{5}\varepsilon$*. Since* $\Phi_v(1), \Phi_v(4), \Phi_v(5) < 0$*, these players cannot be the first to be scheduled in an order feasible for* $\mathbf{x}^\star$*. Moreover observe that, for players* 2 *and* 3 *the maximum fees that they can pay if they are the first to be scheduled in an order feasible for* $\mathbf{x}^\star$ *are* $B(2) = \frac{6}{5}\beta + \frac{4}{5}\varepsilon$ *and* $B(3) = 2\beta + \varepsilon$*, respectively. Thus,* $\Phi_v(2) > B(2)$ *and* $\Phi_v(3) > B(3)$*. Then, there is no order according to which* $\mathbf{\Phi}_v$ *is valid for* $\mathbf{x}^\star$*.*

However, we show that for a subclass of network coordination games it is possible to extend Theorem 6 in order to achieve also Shapley-fairness.

Theorem 7. *Consider a two-strategy SCG such that for each edge* $e = (i, j)$ *we have* $\alpha_i^e(0) = \alpha_i^e(1) = \alpha_j^e(0) = \alpha_j^e(1) = \alpha^e$ *and* $\beta_i^e(0) = \beta_i^e(1) = \beta_j^e(0) = \beta_j^e(1) = \beta^e$*. Suppose, moreover, that in the optimal profile* $x^\star$*, all players adopt the same strategy. Then, a collusion-resistant and Shapley-fair vector of fees order-free valid for* $\mathbf{x}^\star$ *always exists and can be easily computed.*

Proof. Consider the Shapley value $\mathbf{\Phi}_v$ of the cooperative game defined in (3). From Lemma 4, $\mathbf{\Phi}_v$ is in the core and thus, by Lemma 2, it corresponds to a collusion-resistant vector of fees of null cost. Then, we only need to show that this vector of fees is order-free valid for $\mathbf{x}^\star$.

Let $\mathbf{\Phi}_{v_1}, \mathbf{\Phi}_{v_2}, \mathbf{\Phi}_{v_3}$ and $\mathbf{\Phi}_{v_4}$ as in the proof of Lemma 3. Then, by our assumptions on SCG,

$$\Phi_{v_2}(i) = \frac{1}{2}\sum_{e=(i,j)}\left(c_i^e(\overline{x}_i^\star, \overline{x}_j^\star) + c_j^e(\overline{x}_j^\star, \overline{x}_i^\star)\right) = \sum_{e=(i,j)} \alpha^e$$

and

$$\Phi_{v_3}(i) = \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil \sum_{e=(i,j)} \left(c_i^e(\overline{x}_i^\star, x_j^\star) - c_j^e(\overline{x}_j^\star, x_i^\star)\right) = 0.$$

Thus $\Phi_v(i) = p_i(\overline{x}_i^\star) - p_i(x_i^\star) + \sum_{e=(i,j)} \alpha^e + \Phi_{v_4}(i) = B(i) + \Phi_{v_4}(i)$. Then, by (5),

$$\Phi_v(i) \leq B(i) \leq \Gamma^\star(i). \tag{6}$$

for each player i.

Consider now the order π in which we schedule first any player i such that $\Phi_v(i) \geq 0$ and then remaining players. Since $\Gamma^\star(i) \leq \Gamma_\pi(i)$ for each player i, it follows from (6) that the vector of fees $\boldsymbol{\gamma} = \boldsymbol{\Phi}_v$ is inducing $\mathbf{x}^\star$ according to π. Moreover, since $\boldsymbol{\gamma}$ has null cost, the sum of positive fees (the ones payed by players scheduled first) is greater than the sum of negative fees (the ones that should be payed to players scheduled later). Thus, $\boldsymbol{\gamma}$ is also deficit-free according to π. Then, $\boldsymbol{\gamma}$ is a vector of fees valid for $\mathbf{x}^\star$ and, by (6) it is also order-free valid for $\mathbf{x}^\star$. □

3.5 Equal-Fairness, Cost-Fairness and Profile-Fairness

In the previous section we introduced a measure of fairness based on the definition of a specific cooperative game. We also showed that a vector of fees that is fair according to this measure is easy to compute, but only in special cases such a fair vector can be used for inducing the desired profile. On the other hand, the proof of Theorem 3 suggests that many vectors of fees valid (or, by Theorem 6, order-free valid) for the optimal profile of a two-strategy SCG may exist. In such cases, one can wonder which vector of fees has to be chosen. A natural approach would be to choose the fairest one. In this section we suggest some fairness measures that may be adopted.

Let us start by describing some *ideal* vectors of fees. A first example of fair vector of fees $\boldsymbol{\gamma}_\star^f$ is the one in which each player receives the same fee, i.e. $\gamma_\star^f(i) = \frac{1}{n}C(\mathbf{x}^\star)$ for each player i. As an alternative example, we would like to have that fees are such that each player has the same total cost in the game defined by these fees. That is, we would like to consider the fair vector of fees $\boldsymbol{\gamma}_\star^c$ that sets $\gamma_\star^c(i) = \frac{1}{n}\left(C(\mathbf{x}^\star) + \sum_j p_j(x_j^\star)\right) - p_i(x_i^\star)$ for each player i. Yet another example is given by the vector of fees $\boldsymbol{\gamma}_\star^p$ in which each player pays exactly her contribution to $C(\mathbf{x}^\star)$, i.e. $\gamma_\star^p(i) = \sum_{e=(i,j)} c_i^e(x_i^\star, x_j^\star)$. Obviously, there are instances in which these ideal vectors of fees are not valid for $\mathbf{x}^\star$. Then, it seems a good trade-off to ask for a vector of fees valid for $\mathbf{x}^\star$ that is as close as possible to the ideal one. Formally, we say that the vector of fees $\boldsymbol{\gamma}$ is *equal-fair* if it minimizes over all vectors of fees valid for $\mathbf{x}^\star$ the distance $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star^f) = \sum_i \left|\gamma(i) - \gamma_\star^f(i)\right|$. Similarly, $\boldsymbol{\gamma}$ is *cost-fair* if it minimizes the distance $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star^c) = \sum_i |\gamma(i) - \gamma_\star^c(i)|$ and *profile-fair* if it minimizes the distance $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star^p) = \sum_i |\gamma(i) - \gamma_\star^p(i)|$.

Unfortunately, finding the equal-fairest, the cost-fairest or the profile-fairest vector of fees is very hard as highlighted by the following theorem.

Theorem 8. *Given a two-strategy SCG and a constant $K > 0$, it is (strongly) NP-complete to decide if there exists a vector of fees $\boldsymbol{\gamma}$ of null cost inducing the optimal profile $\mathbf{x}^\star$ whose distance $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star^f) \leq K$, even if $\alpha_i^e(0) = \alpha_i^e(1) = \alpha_i^e$ and $\beta_i^e(0) = \beta_i^e(1) = \beta_i^e$ for each player i and each edge e. The claim holds also with $\boldsymbol{\gamma}_\star^c$ or $\boldsymbol{\gamma}_\star^p$ in place of $\boldsymbol{\gamma}_\star^f$.*

Proof. In order to evaluate the membership in NP, it is sufficient to have as a witness a vector of fees $\boldsymbol{\gamma}$ and an order π: then, we can evaluate if $\boldsymbol{\gamma}$ is inducing $\mathbf{x}^\star$ according to π and we can compute the distance of this vector of fees from the ideal one.

For the completeness, we consider a polynomial reduction from the perfectly balanced ordering problem introduced in [BCG+05]. Given an undirected graph $G' = (V', E')$ and an ordering σ on the vertices, let $\delta_\sigma^-(i)$ (resp. $\delta_\sigma^+(i)$) be the number of neighbors of i that are before i (resp. after i) in the ordering σ, and let $\phi_\sigma(i) = |\delta_\sigma^+(i) - \delta_\sigma^-(i)|$ be the *imbalance* of i. An ordering σ is *perfectly balanced* if for any vertex i, $\phi_\sigma(i) \leq 1$. The problem of determining whether a given graph has a perfectly balanced ordering is NP-complete [BCG+05].

Note that $\phi_\sigma(i) \leq 1$ is equivalent to $\phi_\sigma(i) = 0$ if $\delta_{G'}(i) = \delta_\sigma^-(i) + \delta_\sigma^+(i)$ is even, and to $\phi_\sigma(i) = 1$ if $\delta_{G'}(i)$ is odd. So, by denoting by $\mathtt{odd}(G')$ the number of vertices of odd degree in G', an ordering σ is perfectly balanced if and only if $\sum_i \phi_\sigma(i) = \texttt{odd}(G')$. Finally, let P_σ (resp. N_σ) denote the set of vertices where $\delta^+_\sigma(i) - \delta^-_\sigma(i) > 0$ (resp. $\delta^+_\sigma(i) - \delta^-_\sigma(i) \leq 0$). Then we have:

$$\begin{aligned}\sum_{i \in P_\sigma} \phi_\sigma(i) - \sum_{i \in N_\sigma} \phi_\sigma(i) &= \sum_{i \in P_\sigma} (\delta^+_\sigma(i) - \delta^-_\sigma(i)) + \sum_{i \in N_\sigma} (\delta^+_\sigma(i) - \delta^-_\sigma(i)) \\ &= \sum_{i \in V'} (\delta^+_\sigma(i) - \delta^-_\sigma(i)) = 0.\end{aligned}$$

This means that an ordering σ is perfectly balanced if and only if $\sum_{i \in P_\sigma} \phi_\sigma(i) = \texttt{odd}(G')/2$.

We are now able to build the reduction. Consider the following social graph $G = (V, E)$ underlying the network coordination game. G includes: each vertex and each edge of G'; a new vertex 0, adjacent to all vertices in V'; for each vertex $i \in V'$ a new vertex $n+i$ adjacent to i. Fix a constant $\Delta \geq \max_{i \in V'} \delta_{G'}(i)$. The costs are as follows:

- $p_0(1) = M$ and $p_0(0) = \Delta$; $p_i(1) = 0$ and $p_i(0) = \Delta$ for any $i = 1, \ldots, n$; $p_i(1) = p_i(0) = \Delta$ for any $i = n+1, \ldots, 2n$;
- For each edge $e = (i, j)$ with $1 \leq i, j \leq n$, $\alpha^e_i = \alpha^e_j = 0$ and $\beta^e_i = \beta^e_j = 2$;
- For each edge $e = (i, 0)$, $\alpha^e_i = \beta^e_i = \alpha^e_0 = 0$ and $\beta^e_0 = M$;
- For each edge $e = (i, n+i)$, $\alpha^e_i = \alpha^e_{n+i} = \beta^e_{n+i} = 0$ and $\beta^e_i = \Delta - \delta_{G'}(i)$.

Then, by taking M sufficiently large, it turns out that the optimum profile $\mathbf{x}^\star$ occurs when all players play 0. Note that $C(\mathbf{x}^\star) = 0$ and $p_i(x^\star_i) = \Delta$ for any i. Thus, $\gamma^f_\star(i) = \gamma^c_\star(i) = \gamma^p_\star(i) = 0$ for any i. We claim that there exists a vector of fees $\boldsymbol{\gamma}$ of null cost inducing $\mathbf{x}^\star$ such that $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^f_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^c_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^p_\star) \leq \texttt{odd}(G')$ if and only if a perfectly balanced ordering in G' exists.

Suppose first that such an ordering σ exists. Then consider the following schedule π of players in the instance above described: player 0 is scheduled as first, then players $i = n+1, \ldots, 2n$ and finally the other players according to σ. Moreover, we consider the following vector of fees $\boldsymbol{\gamma}$: $\gamma(0) = 0$ and, for any $i = 1, \ldots, n$, $\gamma(n+i) = 0$, $\gamma(i) = 0$ if $\delta_{G'}(i)$ is even, $\gamma(i) = 1$ if $\delta_{G'}(i)$ is odd and $i \in N_\sigma$, $\gamma(i) = -1$ if $\delta_{G'}(i)$ is odd and $i \in P_\sigma$. Note that in σ vertices of even degree have imbalance 0, while vertices of odd degrees have imbalance 1 (half of them in N_σ, half of them in P_σ). Hence, $\texttt{odd}(G')/2$ vertices have $\gamma(i) = -1$ and $\texttt{odd}(G')/2$ vertices have $\gamma(i) = 1$ and then the vector of fees has null cost and $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^f_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^c_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^p_\star) \leq \texttt{odd}(G')$.

Observe that $\Gamma_\pi(0) = M$ and thus $\gamma(0) \leq \Gamma_\pi(0)$. For $i \geq n+1$, $\Gamma_\pi(i) = 0$ and thus $\gamma(i) \leq \Gamma_\pi(i)$. For a player $i = 1, \ldots, n$ with $\delta_{G'}(i)$ even, we have that $\delta^-_\sigma(i) = \delta_{G'}(i)/2$ players j with $1 \leq j \leq n$ are scheduled before i and thus $\Gamma_\pi(i) = 0 \geq \gamma(i)$. Similarly, if $i \in N_\sigma$ and $\delta_{G'}(i)$ is odd then $\delta^-_\sigma(i) = (\delta_{G'}(i)+1)/2$ and thus $\Gamma_\pi(i) = 1 \geq \gamma(i)$. For $i \in P_\sigma$ with $\delta_{G'}(i)$ odd, $\delta^-_\sigma(i) = (\delta_{G'}(i) - 1)/2$ and thus $\Gamma_\pi(i) = -1 \geq \gamma(i)$. Hence, $\boldsymbol{\gamma}$ is inducing $\mathbf{x}^\star$ according to π.

Conversely suppose that we have a vector of fees $\boldsymbol{\gamma}$ of null cost inducing $\mathbf{x}^\star$ with $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^f_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^c_\star) = d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^p_\star) \leq \texttt{odd}(G')$. Let π be the schedule according to which $\boldsymbol{\gamma}$ is inducing $\mathbf{x}^\star$. We can assume that in π player $n+i$ is scheduled before player i for each $i = 1, \ldots, n$. Indeed, suppose this is not the case and in π there exists k for which player $n+k$ is scheduled after player k. Then, since $\boldsymbol{\gamma}$ is inducing according to π, $\gamma(n+k) \leq \Gamma_\pi(n+k) = 0 = \Gamma_{\pi'}(n+k)$, where π' is the order obtained by moving $n+k$ just before k. Moreover, $\gamma(k) \leq \Gamma_\pi(k) \leq \Gamma_{\pi'}(k)$. Finally, for each player $j \neq k, n+k$, $\gamma(j) \leq \Gamma_\pi(j) = \Gamma_{\pi'}(j)$. Thus, $\boldsymbol{\gamma}$ is inducing $\mathbf{x}^\star$ also according to π'.

Then consider the ordering σ obtained from π by removing player 0 and players $n+1, \ldots, 2n$: we claim that σ is a perfectly balanced ordering in G'. Let us consider a player $i \in P_\sigma$. Since $\boldsymbol{\gamma}$ is inducing $\mathbf{x}^\star$ according to π, then $\gamma(i) \leq \Gamma_\pi(i) = 2\delta^-_\sigma(i) - \delta_{G'}(i)$. Thus $\gamma(i) < 0$ and $|\gamma(i)| \geq \delta^+_\sigma(i) - \delta^-_\sigma(i) = \phi_\sigma(i)$. Since $\boldsymbol{\gamma}$ has null cost, then the sum of negative fees in $\boldsymbol{\gamma}$ is $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}^f_\star)/2 \leq \texttt{odd}(G')/2$. Thus $\sum_{i \in P_\sigma} \phi_\sigma(i) \leq \sum_{i \in P_\sigma} |\gamma(i)| \leq \sum_{i:\, \gamma(i) < 0} |\gamma(i)| \leq \texttt{odd}(G')/2$. Hence σ is a perfectly balanced ordering. $\square$

We stress that the hardness result of Theorem 8 holds even if we do not ask for deficit-freeness.

Given this hardness result, it makes sense to ask for a fair vector of fees in a subset of the valid ones. For example, one can ask about the fairest vector of fees among the order-free valid ones. However, as highlighted by Theorem 5, also checking if a vector of fees is order-free is hard. Hence, we are forced to further restrict the set of vectors among which we would like to compute the fairest.

The above sections highlighted that the base value $B(i)$ of a player i is a very important quantity. Indeed, Theorem 6 shows that a vector of fees $\boldsymbol{\gamma}$ such that $\gamma(i) \leq B(i)$ for each player i can be at same time collusion-resistant and order-free valid for the optimal profile of an SCG. Hence, it makes sense to ask for the fairest vector of fees among these ones.

Theorem 9. *The equal-fairest, the cost-fairest and the profile-fairest vectors of fees, among any vector $\boldsymbol{\gamma}$ such that $\gamma(i) \leq B(i)$ for each player i, can be computed in polynomial time.*

Proof. Let us set $\gamma_\star(i) = \frac{1}{n}K + k_i$ for each player i, where $K = C(\mathbf{x}^\star) - \sum_i k_i$, and $(k_i)_{i\in[n]}$ is a vector of player-specific constants. Observe that $\boldsymbol{\gamma}_\star^f$, $\boldsymbol{\gamma}_\star^c$ and $\boldsymbol{\gamma}_\star^p$ can be seen as special forms of $\boldsymbol{\gamma}_\star$ where, respectively, $k_i = 0$, $k_i = p_i(x_i^\star)$ and $k_i = \sum_{e=(i,j)} c_i^e(x_i^\star, x_j^\star)$ for each player i. Thus, the theorem follows by proving that we are able to compute in polynomial time the vector of fees $\boldsymbol{\gamma}$ that minimizes $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star)$.

Let $K_0 = K$ and let us rename players in non-decreasing order of $B(i) - k_i$, i.e., $B(1) - k_1 \leq B(2) - k_2 \leq \ldots \leq B(n) - k_n$. Then, for each $i = 1, \ldots, n$ we set

$$\gamma(i) = \min\left\{B(i), \frac{K_{i-1}}{n-i+1} + k_i\right\}$$

where $K_i = K_{i-1} - \gamma(i) + k_i$. Informally, we try to set $\gamma(i) = \gamma_\star(i) = K/n + k_i$ for each player i. This might be not possible for some player i due to the upper bound $B(i)$. In this case we set $\gamma(i) = B(i)$ but then there is an extra-cost to be shared among players scheduled after (K is updated). Note that $K_{i-1}/(n-i+1) \leq K_i/(n-i)$ (the "shared" part of the cost is non decreasing) and if $\gamma(i) < B(i)$ for some i, then, by recurrence, we get that for each $j \geq i$, $\gamma(j) = K_{i-1}/(n-i+1) + k_j < B(j)$ (the "shared" part of the cost becomes constant).

Obviously, this algorithm runs in time linear in n and $\gamma(i) \leq B(i)$ for each player i. Observe that:

(i) if $\gamma(i) = B(i)$ for each player i, then $\sum_i B(i) = C(\mathbf{x}^\star)$. Indeed, $\gamma(n) \leq K_{n-1} + k_n = K - \sum_{i<n}\gamma(i) + \sum_{i\leq n} k_i$, meaning that $\sum_i B(i) \leq C(\mathbf{x}^\star)$ (while by Lemma 1 $\sum_i B(i) \geq C(\mathbf{x}^\star)$).

(ii) if there is i such that $\frac{K_{i-1}}{n-i+1} + k_i < B(i)$, as said before this will hold also for any $j > i$ and thus $\sum_i \gamma(i) = \sum_i \gamma_\star(i) = C(\mathbf{x}^\star)$. Indeed, $\gamma(n) = K_{n-1} + k_n = K - \sum_{j<n}\gamma(j) + \sum_{j\leq n} k_j$.

Hence, in both cases, $\boldsymbol{\gamma}$ has null cost. (Note that since $\boldsymbol{\gamma}$ sets $\gamma(i) \leq B(i)$ for each player i and has null cost, implies that $\boldsymbol{\gamma}$ is order-free valid for $\mathbf{x}^\star$ and collusion-resistant.)

To complete the proof we show that the vector $\boldsymbol{\gamma}$ of fees returned by the algorithm minimizes $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star)$. In fact, we show that for each player i, $|\gamma(i) - \gamma_\star(i)|$ can be reduced only by increasing $|\gamma(j) - \gamma_\star(j)|$ for some $j \neq i$ such that $|\gamma(j) - \gamma_\star(j)| \geq |\gamma(i) - \gamma_\star(i)|$ and then increasing the distance $d(\boldsymbol{\gamma}, \boldsymbol{\gamma}_\star)$. We distinguish two cases: if $\gamma(i) < \gamma_\star(i)$, then $\gamma(i) = B(i)$ and thus $|\gamma(i) - \gamma_\star(i)|$ cannot be reduced at all. If $\gamma(i) > \gamma^\star(i)$, then we consider a player j. Note that, in order to move some fraction of $\gamma(i)$ towards $\gamma(j)$, $\gamma(j)$ should be strictly less than $B(j)$. In particular, if $M = K_{j-1}/(n-j+1) \geq \frac{1}{n}K$ is the shared part of the cost at iteration j, then $\gamma(j) = M + k_j$. Moreover, $\gamma(\ell) \leq M + k_\ell$ for any player $\ell < j$ (the shared part of the cost is non decreasing) and $\gamma(\ell) = M + k_\ell$ for a player $\ell > j$ (the shared part of the cost becomes constant). Thus, $|\gamma(j) - \gamma_\star(j)| = M - \frac{1}{n}K$, and $\gamma(i) \leq M + k_i$ meaning that $|\gamma(i) - \gamma_\star(i)| \leq M - \frac{1}{n}K$. □

4 Extensions

4.1 To Other Social Network Games

Even if two-strategy SCGs model several interesting phenomena, it would be interesting to understand what happens in other social network games. Unfortunately, in this section we will show that computing the optimal profile for these extensions may be difficult, making less interesting the design mechanisms for inducing such a profile.

We start by considering the possibility that the edge game $\mathcal{G}_e$ is not a coordination game. Then we have the following lemma.

Lemma 5. *Computing the optimal profile in a social network game is (strongly) NP-hard even if each player has at most two strategies, for every edge game $\mathcal{G}_e$ there is exactly one player preferring to disagreeing instead of agreeing and all preference costs are null.*

Proof. We consider a polynomial time reduction from the maximum cut problem [GJ79].

Consider a graph $G = (V, E)$. We consider a social network game defined upon G and two strategies 0 and 1 per player. We assume null preference costs for each player and for each edge $e = (i, j)$ player i pays 0 both for an agreement and for a disagreement, whereas player j pays 0 for a disagreement and 1 for an agreement.

Consider the following one-to-one correspondence between a cut (L, R) of G and a profile $\mathbf{x}$ in the corresponding game: $x_i = 0$ if $i \in L$ and $x_i = 1$ otherwise. The total cost of this profile $\mathbf{x}$ is $|E(L, L)| + |E(R, R)| = |E| - |E(L, R)|$ where $E(A, B)$ represents the set of edges $e = (i, j)$ such that $i \in A$ and $j \in B$. Hence, the minimum cost profile coincides with the maximum cut in the graph G. □

Another possible extension would be to consider SCGs in which the players have more than two strategies available. The next lemma shows that also for this extension the optimal profile is hard to compute.

Lemma 6. *Computing the optimal profile in an SCG in which the players have more than two strategies is (strongly) NP-hard, even if for each player i and each edge game $\mathcal{G}_e$ played by i there is one value β_i^e such that any agreement cost is null and any disagreement cost is β_i^e, regardless of the strategy adopted by i.*

Proof. We consider a polynomial time reduction from the minimum k-cut problem with specified vertices. In this problem, we have an edge weighted undirected graph $G = (V, E)$ (with $W_e \geq 0$ for any edge e), an integer k and k vertices $u_1, u_2, \dots, u_k$. The goal is to find a partition of V into k subsets $V_1, V_2, \dots, V_k$ such that $u_i \in V_i$ for any i and the sum of the weights of edges whose endpoints are in two different parts is minimized. This problem is NP-hard for any $k \geq 3$ [GH88]. Without loss of generality, we suppose that the graph is complete (by putting edges with weight 0) and the weight of edges between u_i and u_j is 0.

We consider a social network game with k strategies $\{1, 2, \dots, k\}$. To each vertex in $V' = V \setminus \{u_1, u_2, \cdots, u_k\}$ corresponds a player. The costs are as follows:

- If two players i and j play the same strategy it induces a cost 0. Otherwise it induces a cost $W_{(i,j)}/2$ for both i and j.
- The preference cost of player i playing $x_i \in \{1, 2, \dots, k\}$ is $p_i(x_i) = \sum_{j \neq x_i} W_{(i,u_j)}$.

By identifying the set of players playing strategy s with the set $V_s \setminus \{u_s\}$ in the minimum k-cut problem, we have a one-to-one correspondence between feasible solutions. The total cost of a profile is

$$\sum_{i \in V'} p_i(x_i) + \sum_{\substack{e=(i,j) \\ x_i \neq x_j}} W_e = \sum_{i \in V'} \sum_{j \neq x_i} W_{(i,u_j)} + \sum_{\substack{e=(i,j) \\ x_i \neq x_j}} W_e$$

which is the value of the corresponding k-cut. □

4.2 To Non-Optimal Profiles

Given these negative results, one may be interested in inducing a non-optimal profile of a social network game as, for example, the output of an approximation algorithm or a heuristic. Unfortunately, the next lemma shows that deciding if a frugal incentive-compatible best-response mechanism for this profile exists can be hard even for two-strategy games. We stress that the hardness result holds even if we relax the constraint given in (1) and we do not ask for deficit-freeness.

Lemma 7. *Given a two strategy social network game and a profile $\mathbf{x}$, it is (strongly) NP-complete to decide if a vector of fees $\boldsymbol{\gamma}$ inducing $\mathbf{x}$ with cost at most 0, even if all preference costs are null and for each player i and each edge game $\mathcal{G}_e$ played by i there are two values $\alpha_i^e \leq \beta_i^e$ such that any agreement costs α_i^e and any disagreement costs β_i^e, regardless of the strategy adopted by i.*

Proof. For verifying the membership to NP we use an order π as a witness. Then, it is sufficient to verify that γ is inducing $\mathbf{x}$ according to π and that $\sum_i \gamma(i) \geq C(\mathbf{x})$. As for the completeness, we will show a polynomial reduction from the problem of deciding if a feedback arc set of size at most M exists in a direct graph $D = (U, H)$. The reduction is similar to the one used for proving Theorem 5.

Given an instance of the feedback arc set problem, we build an SCG as follows: there is a player for every vertex in U and a new player, that we name $i_\star$. Each player i has strategies 0 and 1 and $p_i(0) = p_i(1) = 0$. As for the edges of the social graph, consider the graph $D^\star = (U, H^\star)$ obtained by completing D (see the proof of Theorem 5). The social network underlying the game is given by the graph $G = (U \cup \{i_\star\}, E)$ where E contains any edge between i and j, if these two profiles are adjacent on $H^\star$, and an edge $(i_\star, i)$ for any i. As for the game $\mathcal{G}_e$ played on the edge $e = (i, j)$ with $i, j \in U$, we set $\alpha_i^e = \alpha_j^e = 0$, $\beta_i^e = W_{(j,i)}$ and $\beta_j^e = W_{(i,j)}$. As for the game $\mathcal{G}_e$ played on the edge $e = (i_\star, j)$, we set $\alpha_{i_\star}^e = \alpha_j^e = \beta_j^e = 0$ and $\beta_{i_\star}^e = \frac{|H|-M}{|U|}$. Finally, consider a profile $\mathbf{x}$ such that $x_{i_\star} = 1$ and the remaining players play 0. Note that $C(\mathbf{x}) = |H| - M$.

We will show that if a feedback arc set of size at most M exists, then there is a vector of fees inducing $\mathbf{x}$ with cost at most 0. Consider indeed the DAG $D' = (U, H')$ resulting from the elimination of the feedback arc set and let us build an order π as follows: we first schedule players from V in a topological ordering of vertices in D', then we schedule the player $i_\star$. Observe that, for each player $u \in V$, $\Gamma_\pi(u)$ is exactly the number of ingoing edges in D'. Moreover, $\Gamma_\pi(i_\star) = 0$. Consider now the vector of fees $\boldsymbol{\gamma}$ that sets $\gamma(k) = \Gamma_\pi(k)$ for each player k. Obviously, $\boldsymbol{\gamma}$ is inducing according to π. Moreover $\sum_{u \in U} \gamma(u) = \sum_{u \in U} \Gamma_\pi(u) = |H'| \geq |H| - M$. Thus, $\boldsymbol{\gamma}$ has cost at most 0.

We conclude the proof, by showing that if a vector of fees $\boldsymbol{\gamma}$ inducing $\mathbf{x}$ with cost at most 0, then there is a feedback arc set of size at most M in D. Let π be the order according to which $\boldsymbol{\gamma}$ is inducing $\mathbf{x}$. Observe that $\gamma(i_\star) \leq \Gamma_\pi(i_\star) = 0$ regardless of the position in which $i_\star$ is scheduled. Then, since $\boldsymbol{\gamma}$ has cost at most 0, $\sum_{u \in U} \gamma(u) \geq |H| - M$. Moreover, we can build a DAG $D' = (U, H')$ from π and the graph D by adding to H' only edges $(u, v) \in H$ such that u is scheduled before v in π. Observe that, for each $u \in U$, $\Gamma_\pi(u) = \sum_{v \in N_\pi(u)} W_{(v,u)}$ is exactly the number of ingoing edges in D'. Thus, $\sum_{u \in U} \Gamma_\pi(u) = |H'|$. Since $\sum_{u \in U} \Gamma_\pi(u) \geq \sum_{u \in U} \gamma(u)$, we have $|H| - |H'| \leq M$. That is, a DAG has been obtained by D by deleting at most M edges and thus a feedback arc set of size at most M exists. $\square$

Given this result, it makes sense to think about designing incentive-compatible best-response mechanisms for inducing a non-optimal profile $\mathbf{x}$ in which a designer has a cost of at most K. Unfortunately, it is immediate to extend the previous theorem in order to rule out this possibility as well.

5 Conclusions and Open Problems

The focus of this work is the design of mechanisms through which an authority may influence the bargaining among components of a social network for inducing optimal states. Our mechanism adopts a "dynamical approach", that is tends to modify the game in order to have that natural dynamics converge to the target state. We think that this approach can be promising for the design of mechanisms in many different settings.

The main results of this work refer to the special case of inducing the optimal profile of two-strategy social coordination games. Indeed, we proved that computing the optimal profile for other kind of games may be hard. However, it should be interesting to understand if a mechanism can be designed for special instances (small instances or special network topologies) for which the optimum is easy to compute.

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A Cooperative Cost Games

A *cooperative cost game* (or, simply, a *cost game*) is a pair $([n], v)$, where $[n]$ is the set of players and v is the *characteristic function*, assigning to each *coalition* $L \subseteq [n]$, a positive real number $v(L) \in \mathbb{R}$ representing the cost of L, with $v(\emptyset) = 0$ by convention. When clear from the context we omit the reference to the set of players and we identify a cost game $([n], v)$ with the corresponding characteristic function v.

A *one-point solution* (or simply a *solution*) for a cost game v is a vector $\boldsymbol{\psi}_v = (\psi_v(i))_{i\in[n]}$. A solution $\boldsymbol{\psi}_v$ is *efficient* if $\sum_i \psi_v(i) = v([n])$. The *core* of a cost game v is defined as the set of efficient solutions $\mathbf{x} \in \mathbb{R}^{[n]}$ for which no coalition has an incentive to leave the grand coalition $[n]$. That is

$$\texttt{core}(v) = \left\{ \mathbf{x} \in \mathbb{R}^{[n]} \colon \sum_{i\in L} x_i \leq v(L) \quad \forall L \subset [n], L \neq \emptyset; \sum_{i\in[n]} x_i = v([n]) \right\}.$$

B Some Useful Property of Shapley Value

We recall some nice properties of the Shapley value of a cost game v: *efficiency*, i.e. $\sum_{i\in[n]} \Phi_v(i) = v([n])$; *symmetry*, i.e. if $v(L \cup i) = v(L \cup j)$ for all $L \subset [n]$ such that $i, j \in [n] \setminus L$, then $\Phi_v(i) = \Phi_v(j)$; *dummy player property*, i.e. if $i \in [n]$ is such that $v(L \cup i) - v(L) = 0$ for all $L \subseteq [n]$, then $\Phi_v(i) = 0$; *additivity*, i.e. $\boldsymbol{\Phi}_{v_1} + \boldsymbol{\Phi}_{v_2} = \boldsymbol{\Phi}_{v_1+v_2}$ for each v_1, v_2. It is well known that the Shapley value is the only solution that satisfies these four properties [Sha53].

Given a coalition L, $L \neq \emptyset$, and $k \in \mathbb{R}$, the game defined by the characteristic function u_L^k such that $u_L^k(T) = k$ if $L \subseteq T$, and $u_L^k(T) = 0$, otherwise, for each $T \subseteq [n]$, is called *unanimity game* on L. By the properties of efficiency, symmetry and dummy player, it immediately follows that the Shapley value of an unanimity game is $\Phi_{u_L^k}(i) = \frac{k}{\ell}$ if $i \in L$, and $\Phi_{u_L^k}(i) = 0$, otherwise. Observe that $\boldsymbol{\Phi}_{u_L^k}$ is in the core of u_L^k.